Predictive information: From definition to applications to biological systems

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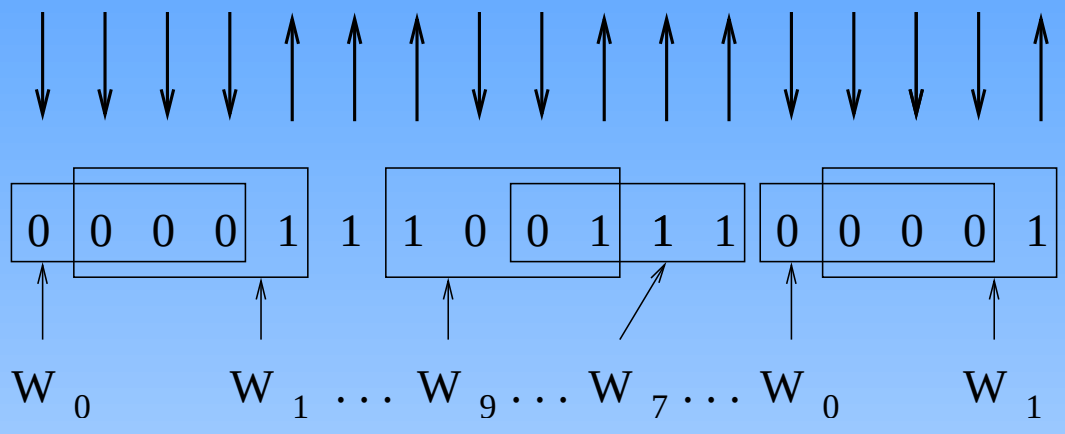
Outline

- A curious observation.
- Quantifying predictability.
- Predictability and optimization in sensory information processing.
- Learning and predictive information.
- Testing models used by animals.
- Bonus material.

Entropy of words in a spin chain



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$W_1 = 0\ 0\ 0$

$W_2 = 0\ 0\ 1$

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$W_{15} = 1\ 1\ 1$

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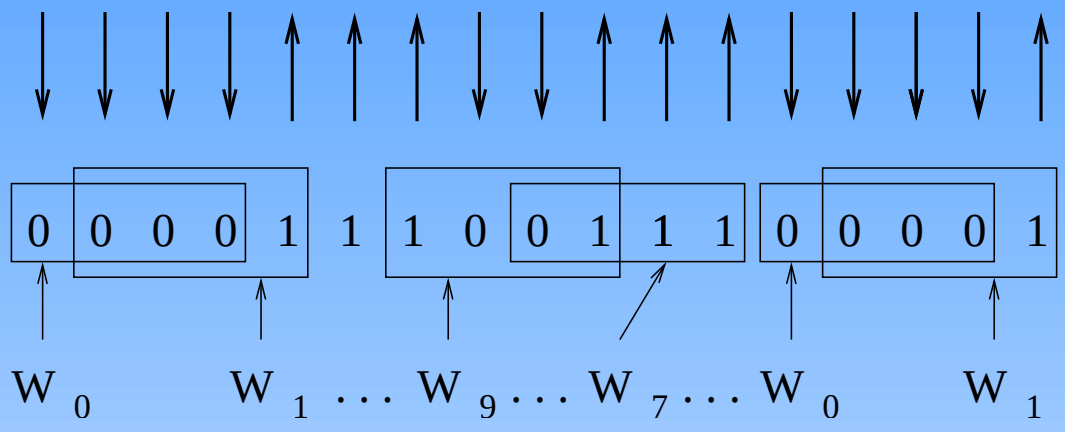

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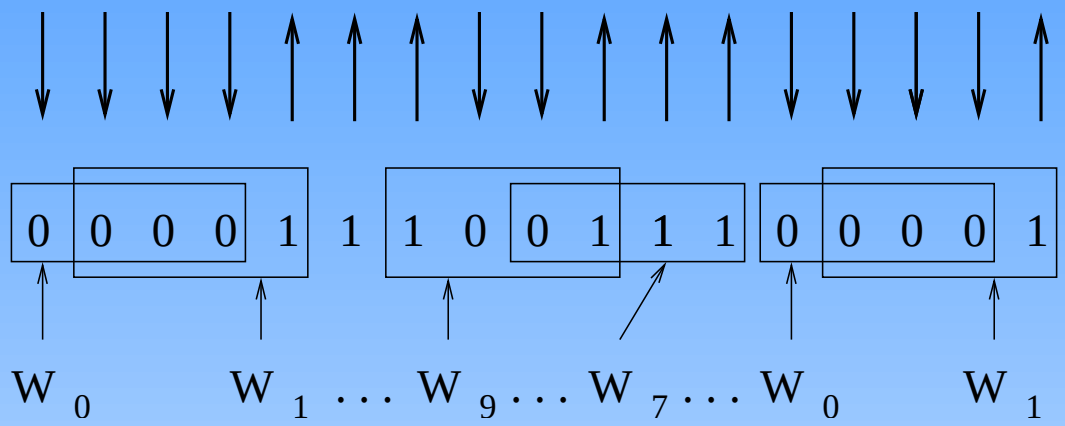

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. . .
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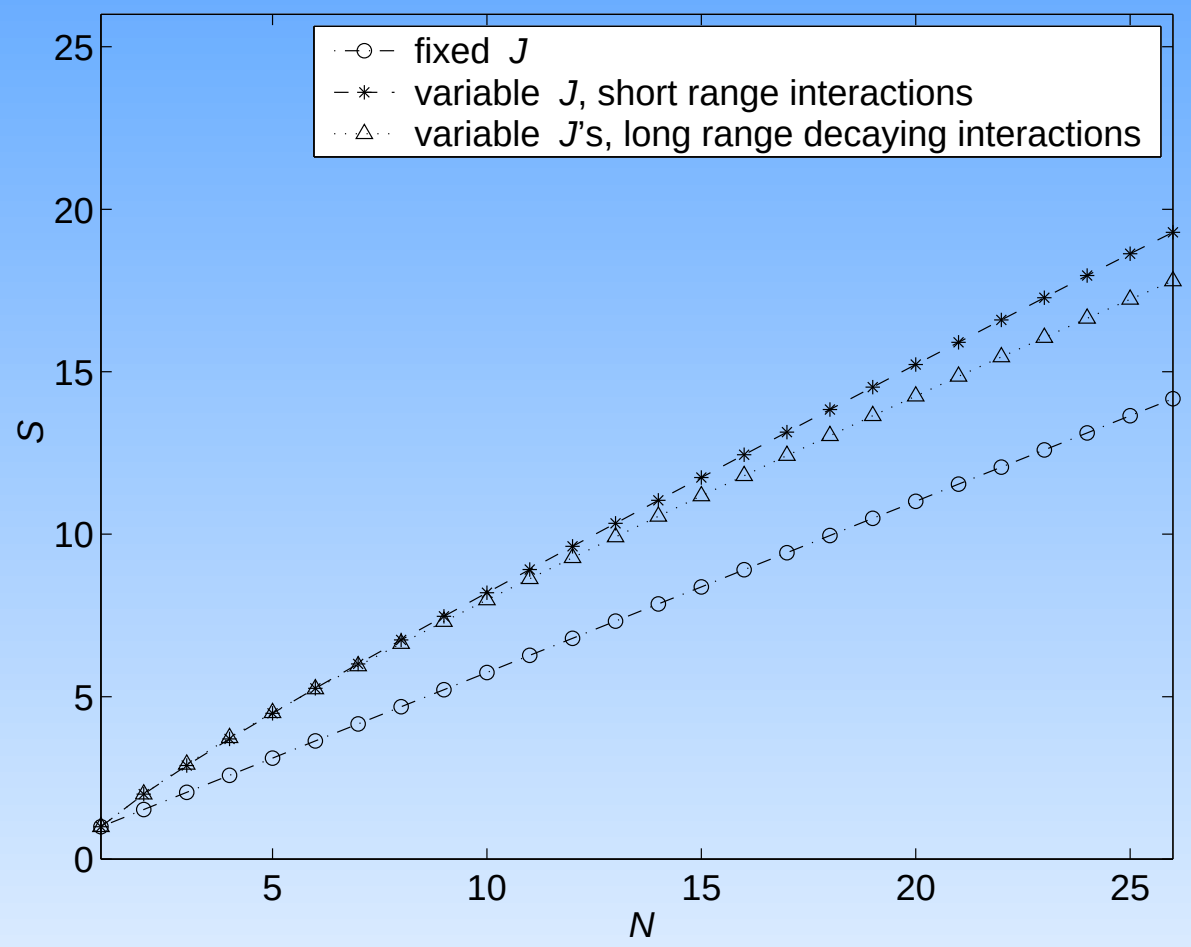
$$S(N) = -\sum_{k=0}^{2^N-1} P_N(W_k)\log_2 P_N(W_k)$$

For this chain, $P(W_0) = P(W_1) = P(W_3) = P(W_7) = P(W_{12}) = P(W_{14}) = 2$, $P(W_8) = P(W_9) = 1$, and all other frequencies (probabilities) are zero. Thus, $S(4) \approx 2.95$ bits.

Entropy of 3 generated chains

- $J_{\mathrm{ij}} = \delta_{\mathrm{i,j+1}}$
- $J_{\mathrm{ij}} = J_0\,\delta_{\mathrm{i,j+1}}$, J_0 is taken at random from $\mathcal{N}(0,1)$ every 400000 spins
- J_{ij} is taken at random from $\mathcal{N}(0,\frac{1}{\mathrm{i-j}})$ every 400000 spins

$1 \cdot 10^9$ spins total.



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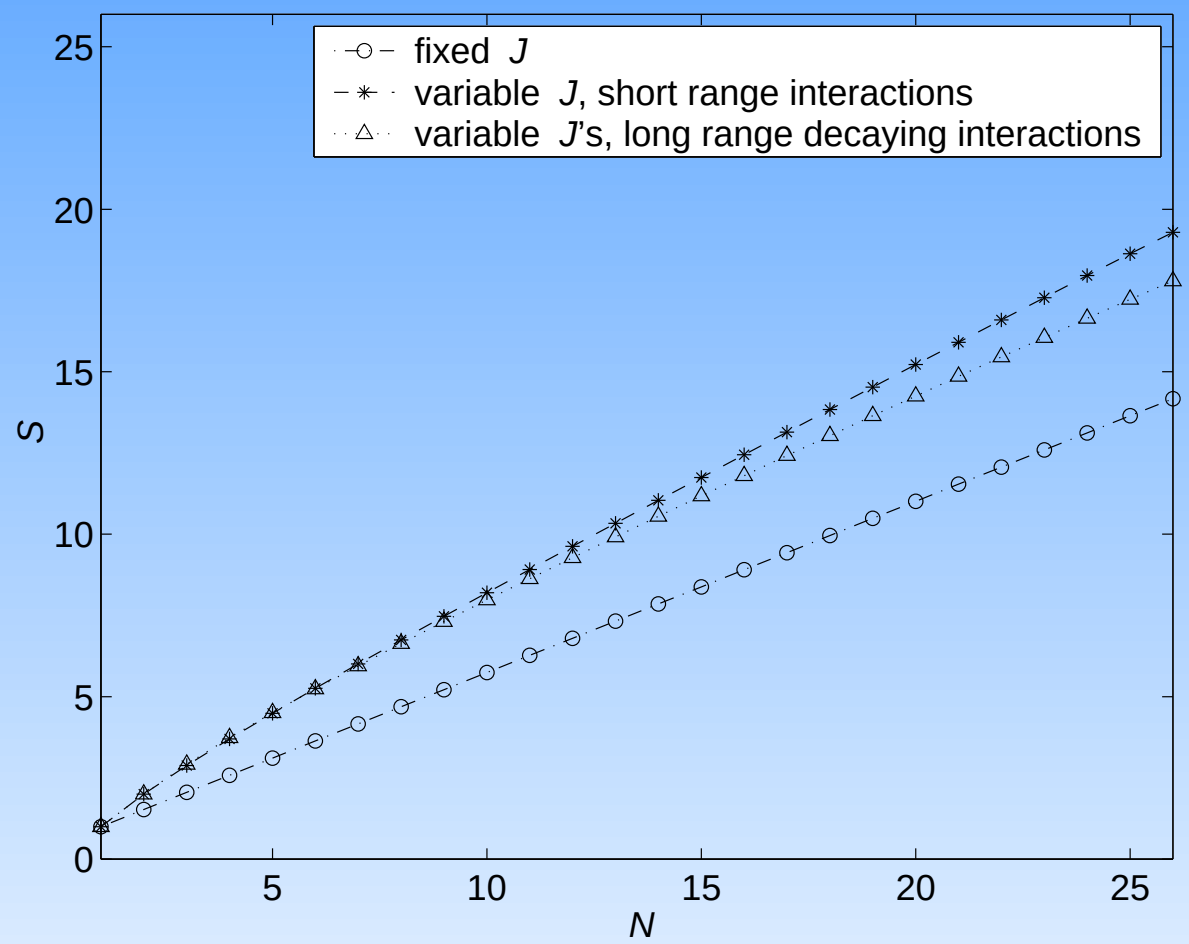
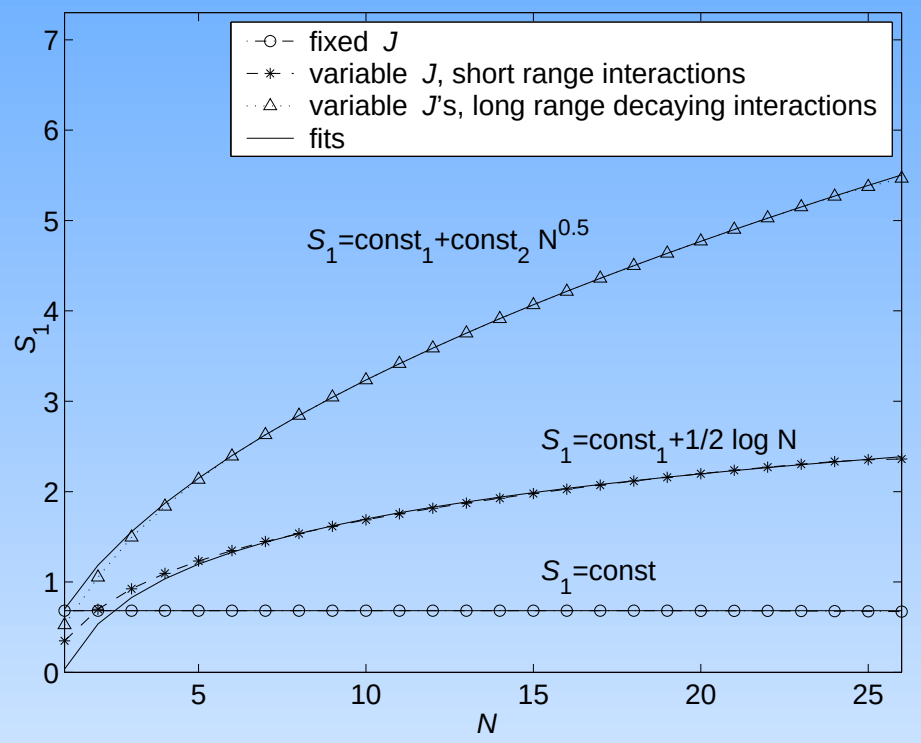


Entropy is extensive!

It shows no distinction between the cases.

Subextensive component of the entropy

. . . shows a qualitative distinction between the cases!



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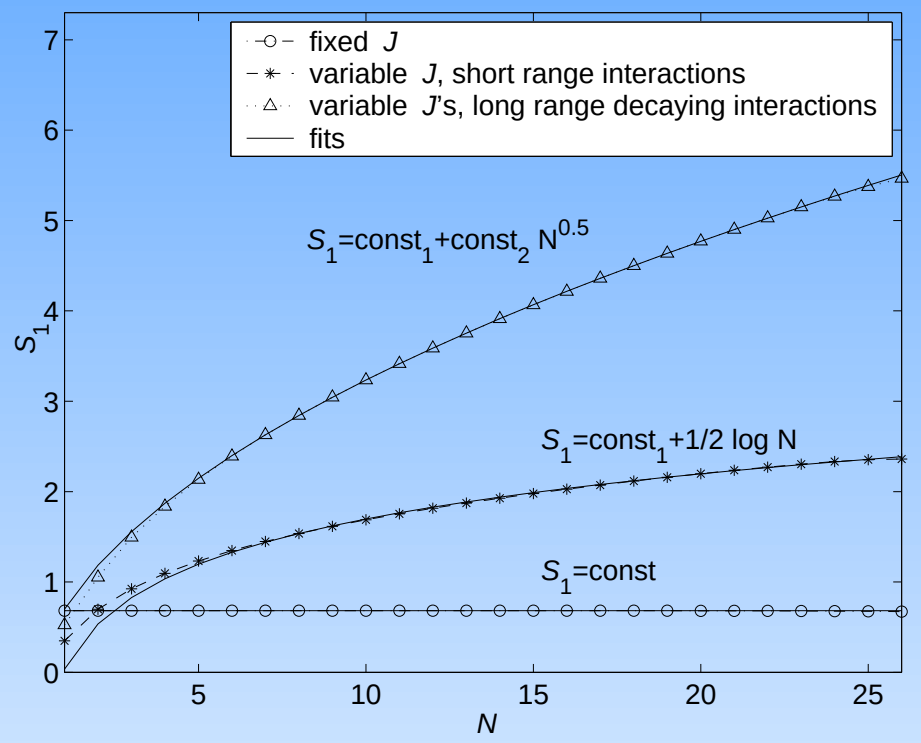

Other examples:

const periodic sequences, chaotic sequences (finite correlation length)

log systems at phase transitions, or at the onset of chaos (divergent correlation length)

power natural texts, DNA sequences, (possibly) some exotic transitions, (many divergent correlation lengths)

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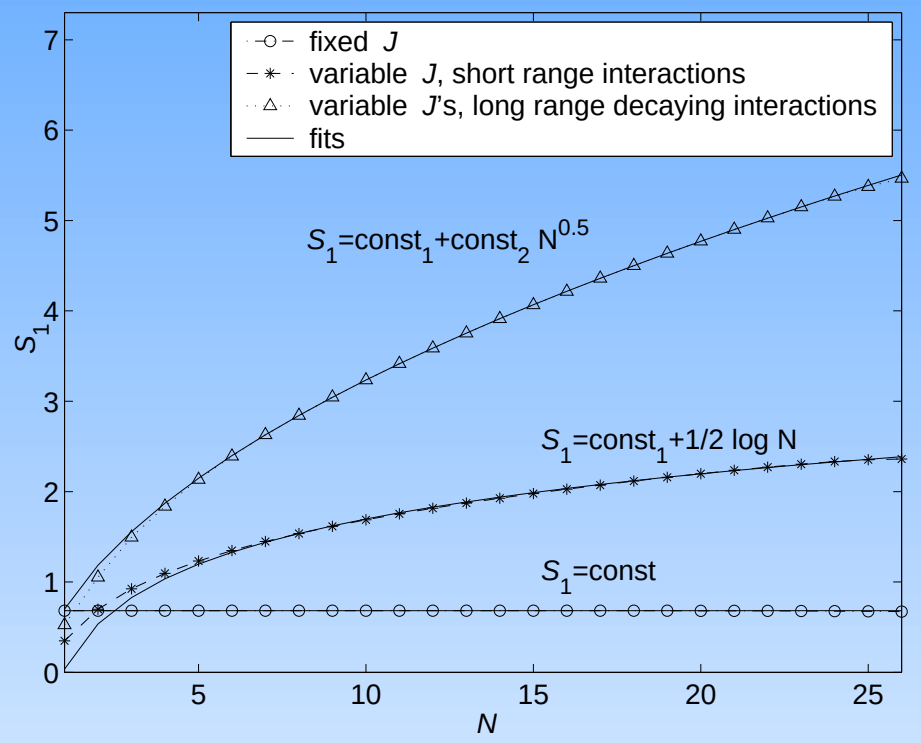

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- Entropy density or channel capacity do not distinguish these cases.
- Theory of phase transitions may not distinguish between the last two cases.
- Complexity of underlying dynamics intuitively increases from **const** to **power**.

Objectives

- unified description of complexity and learning
- make distinction between useful and unusable data
- do this using physical quantities
- understand models used by organisms to represent the world
- understand biological designs by means of optimization principles

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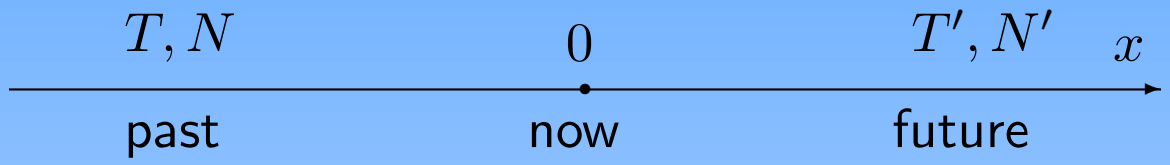
- optimizing predictive information may be the design principle

Quantifying predictability

Information theory: non-metric, universal way to quantify learning



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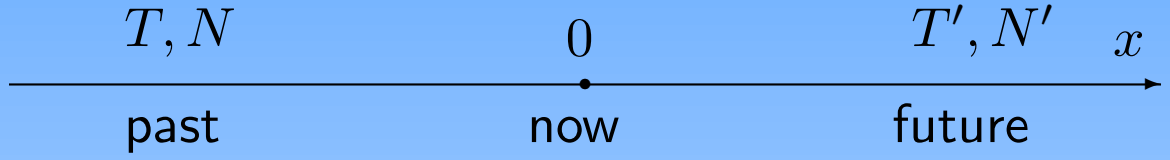

$$\begin{aligned}\mathcal{I}_{\text{pred}}(T, T') &= \left\langle \log_2 \left[\frac{P(x_{\text{future}}|x_{\text{past}})}{P(x_{\text{future}})} \right] \right\rangle \\ &= S(T) + S(T') - S(T + T')\end{aligned}$$

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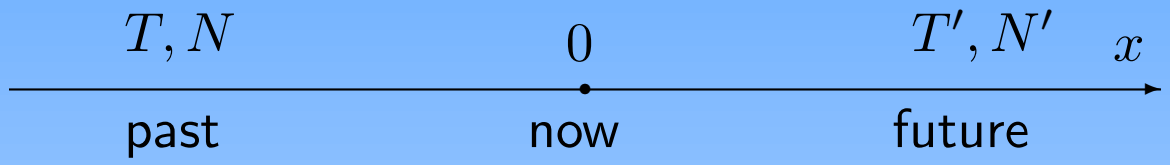

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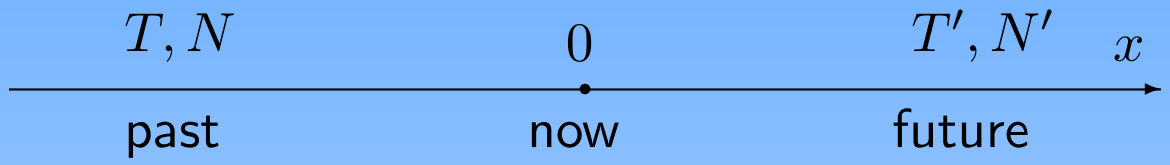

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$$I_{\text{pred}}(T) \equiv \mathcal{I}_{\text{pred}}(T, \infty) = S_1(T)$$

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- diminishing returns, $\lim_{T\to\infty} \frac{I_{\rm pred}(T)}{S(T)} = 0$
- prediction and postdiction are symmetric
- it relates to and generalizes many relevant quantities
  - learning: universal learning curves
  - complexity: complexity measures
  - coding: model coding length

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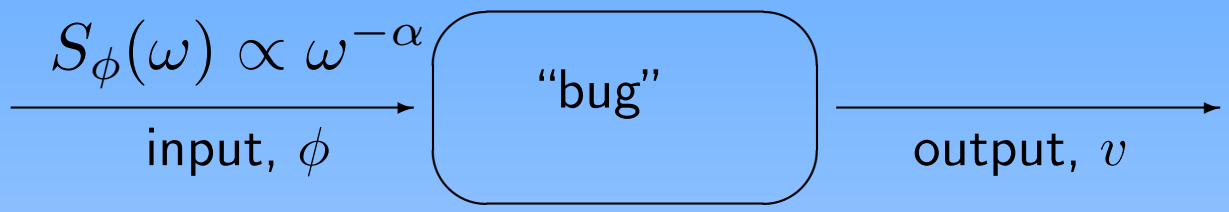
$\lim_{N\to\infty} I_{\rm pred} = {\rm const} \times N^{\xi}$ learning more features as N grows

- learning continuous densities
- language
- some critical phenomena (wetting transitions)
- not well studied

I_{pred} optimization in biology



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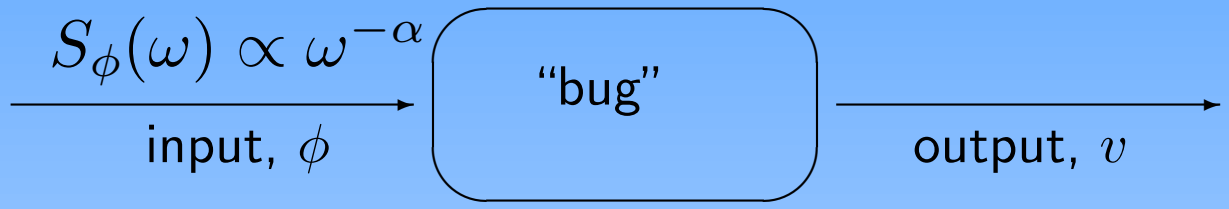

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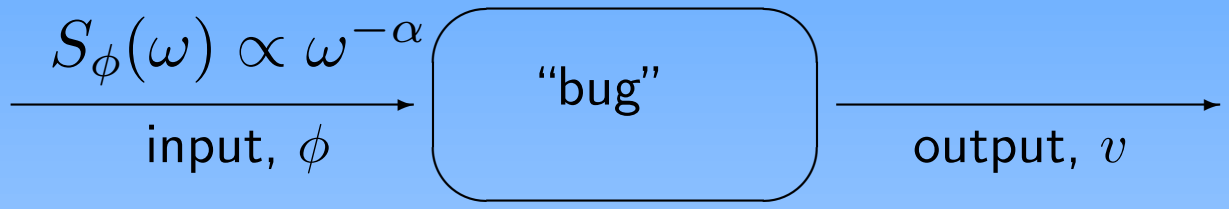

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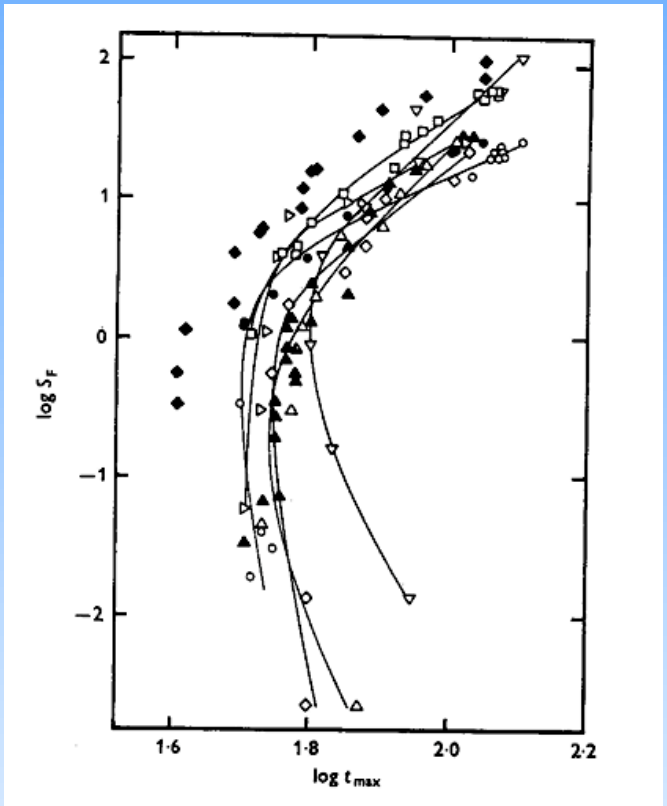
$$\mathcal{I}([\phi],[v]) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} \frac{d\omega}{2\pi} \log\left(1 + \frac{S_\phi(\omega)}{1/I_0}\right)$$

Maximization w.r.t. τ is meaningless.

Predictive information maximization



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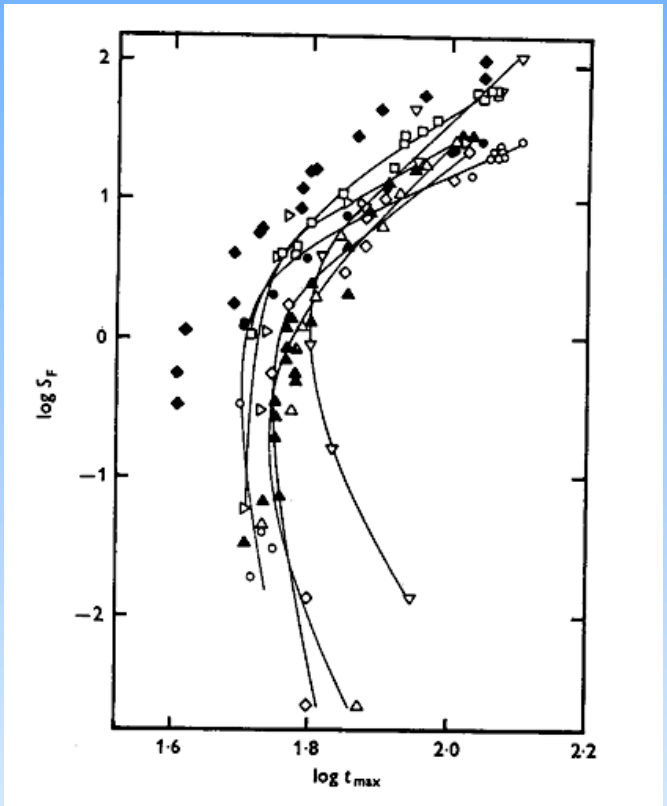

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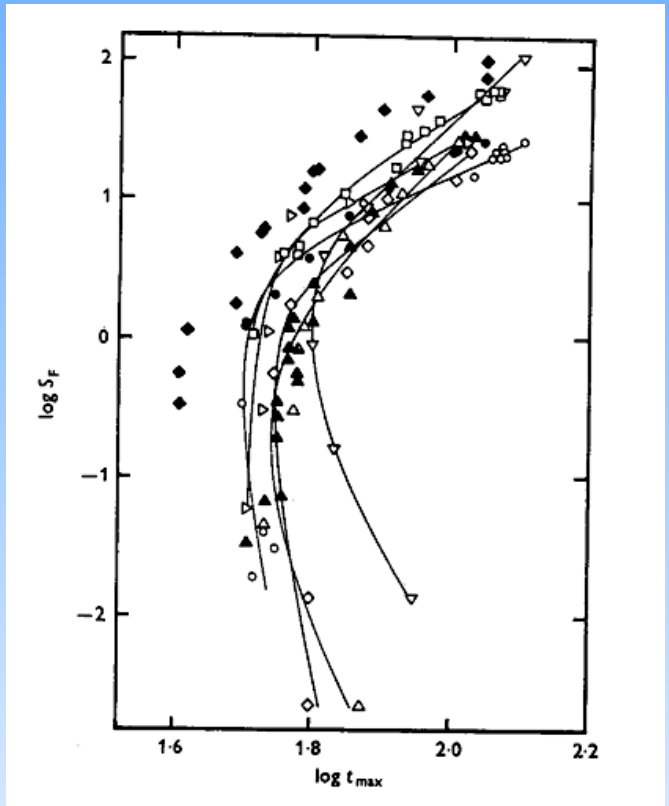

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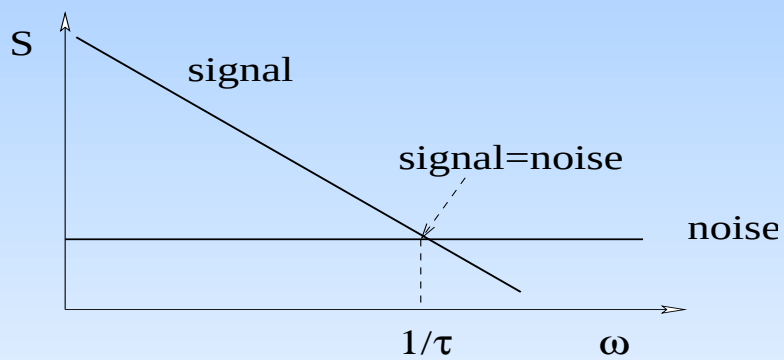
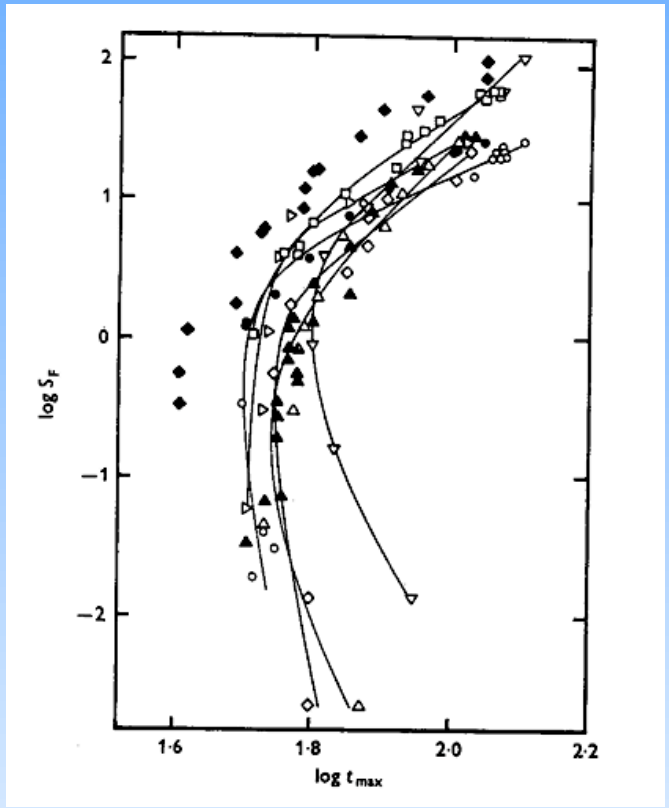
Solution – matching filter: $\tau = I_0^{-1/\alpha}$.





Specific examples: problem setup

$Q(\vec{x}|\boldsymbol{\alpha})$ p. d. f. for $\vec{x}$ parameterized by unknown parameters $\boldsymbol{\alpha}$

$\dim \boldsymbol{\alpha} = K$ dimensionality of $\boldsymbol{\alpha}$, may be infinite

$\mathcal{P}(\boldsymbol{\alpha})$ prior distribution of parameters

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\begin{aligned}
P(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{\mathrm{N}}|\boldsymbol{\alpha}) &= \prod_{\mathrm{i}=1}^{\mathrm{N}} Q(\vec{x}_{\mathrm{i}}|\boldsymbol{\alpha}) \\
P(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{\mathrm{N}}) &= \int d^K\alpha \mathcal{P}(\boldsymbol{\alpha}) \prod_{\mathrm{i}=1}^{\mathrm{N}} Q(\vec{x}_{\mathrm{i}}|\boldsymbol{\alpha}) \\
S(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_{\mathrm{N}}) &\equiv S(N) \\
&= -\int d\vec{x}_1 \cdots d\vec{x}_{\mathrm{N}}\, P(\{\vec{x}_{\mathrm{i}}\}) \log_2 P(\{\vec{x}_{\mathrm{i}}\})
\end{aligned}
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Separating the terms

$$\mathcal{S}_0 = \int d^K\alpha\mathcal{P}(\boldsymbol{\alpha})\left[-\int d\vec{x}\,Q(\vec{x}|\boldsymbol{\alpha})\log_2 Q(\vec{x}|\boldsymbol{\alpha})\right]$$

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$$\mathcal{E}_N \equiv \frac{1}{N} \sum_i \log \left[\frac{Q(\vec{x}_{\mathrm{i}}|\bar{\boldsymbol{\alpha}})}{Q(\vec{x}_{\mathrm{i}}|\boldsymbol{\alpha})} \right] \xrightarrow{\text{anneal}} \int d\vec{x}\, Q(\vec{x}|\bar{\boldsymbol{\alpha}}) \log \frac{Q(\vec{x}|\bar{\boldsymbol{\alpha}})}{Q(\vec{x}|\boldsymbol{\alpha})}$$

Annealed approximation (almost) always works.

Density of states

$$
\begin{aligned}
Z(\bar{\boldsymbol{\alpha}}; N) &= \int d\epsilon\, \rho(\epsilon; \bar{\boldsymbol{\alpha}}) \exp[-N\epsilon] \\
\rho(\epsilon; \bar{\boldsymbol{\alpha}}) &= \int d^K\alpha\, \mathcal{P}(\boldsymbol{\alpha}) \delta[\epsilon - D_{\mathrm{KL}}(\bar{\boldsymbol{\alpha}} || \boldsymbol{\alpha})]
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The density ρ could be very different for different targets.

Learning is annealing at decreasing temperature.

Nonzero $\rho \Longrightarrow$ consistency in learning.

Density at $\epsilon \to 0$, $I_{\rm pred}$, and learning

Occam factor, generalization error, prediction error, fluctuation determinant:

$$\mathcal{D}(\bar{\boldsymbol{\alpha}}; N) \approx -\log \int d\epsilon\, \rho(\epsilon; \bar{\boldsymbol{\alpha}}) \mathrm{e}^{-N\epsilon}$$

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Universal learning curves:

$$\begin{aligned} \Lambda(\bar{\boldsymbol{\alpha}}; N) &\equiv D_{\rm KL}(\bar{\boldsymbol{\alpha}} || \boldsymbol{\alpha}_{\rm est}) \approx \frac{d\mathcal{D}(\bar{\boldsymbol{\alpha}}; N)}{dN} \\ \Lambda(N) &\equiv \int d\bar{\boldsymbol{\alpha}}\, \mathcal{P}(\bar{\boldsymbol{\alpha}}) \Lambda(\bar{\boldsymbol{\alpha}}; N) \approx \frac{dI_{\rm pred}}{dN} \end{aligned}$$

Finite number of states and finite I_{pred}

$$\rho(\epsilon; a_1) = \sum_{i=1}^{M} \mathcal{P}_i \delta(d_i - \epsilon)$$

Finite number of states and finite $I_{\rm pred}$

$$\begin{aligned}
\rho(\epsilon; a_1) &= \sum_{i=1}^{M} \mathcal{P}_i \delta(d_i - \epsilon) \\
\mathcal{D}(a_1; N) &= C_1 - C_2 \exp[-NC_3] \\
\Lambda(a_1; N) &\approx C_2 C_3 \exp[-NC_3]
\end{aligned}$$

$I_{\rm pred}$ saturates as $N \to \infty$

Power–law density function

$$\rho(\epsilon \to 0; \bar{\boldsymbol{\alpha}}) \approx A(\bar{\boldsymbol{\alpha}})\epsilon^{(d-2)/2}$$

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Example: *sound* finite parameter models, $\dim \boldsymbol{\alpha} = d$.

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Speed of approach to this asymptotics is rarely investigated.

Another example

Learning $Q(\vec{x}_1 \cdots \vec{x}_{\mathrm{N}}|\boldsymbol{\alpha})$, a finite parameter Markov process with long range intrinsic correlations such that

$$\begin{aligned} S\left[\{\vec{x}_{\mathrm{i}}\}|\boldsymbol{\alpha}\right] &\equiv -\int d^N\vec{x}\, Q(\{\vec{x}_{\mathrm{i}}\}|\boldsymbol{\alpha})\, \log_2 Q(\{\vec{x}_{\mathrm{i}}\}|\boldsymbol{\alpha}) \\ &\to N\mathcal{S}_0 + \mathcal{S}_0^*; \qquad \mathcal{S}_0^* = \frac{K'}{2}\log_2 N \end{aligned}$$

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Predictive information does not distinguish predictability coming from unknown parameters and from intrinsic long–range correlations. This is similar to describing physical systems with correlations using order parameters.

Essential singularity in the density

$$\rho(\epsilon \to 0; \bar{\boldsymbol{\alpha}}) \approx A(\bar{\boldsymbol{\alpha}}) \exp\left[-\frac{B(\bar{\boldsymbol{\alpha}})}{\epsilon^{\mu}}\right], \quad \mu > 0$$

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- finite parameter model with increasing number of parameters $K \sim N^{\mu/(\mu+1)}$; $S_1(N) \sim N^{\mu/\mu+1}$, not $S_1(N) \sim \frac{N^{\mu/\mu+1}}{2} \log N$
- as $\mu \to \infty$ complexity grows and then vanishes to the leading order when $S_1^{(\mathrm{a})}$ becomes extensive

Example of the power–law $I_{\rm pred}$

Learning a smooth nonparameteric density $Q(x) = 1/l_0 \mathrm{e}^{-\phi(x)}$, $x \in [0, L]$ (Bialek, Callan, and Strong 1996), Complete model.

$$\mathcal{P}[\phi(x)] = \frac{1}{\mathcal{Z}} \exp\left[-\frac{l}{2}\int dx \left(\frac{\partial\phi}{\partial x}\right)^2\right] \delta\left[\frac{1}{l_0}\int dx\, \mathrm{e}^{-\phi(x)} - 1\right]$$

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- increasing number of "effective parameters" (bins) of adaptive size $\sim \sqrt{l/NQ(x)}$
- heuristic arguments for the dimensionality ζ and the smoothness exponent η give $S_1(N) \sim N^{\zeta/2\eta}$ — demonstrates a crossover from complexity to randomness

Nonuniform $\mathcal{D}$

Nested finite parameter models, $r = 1 \ldots \infty$, $K = K(r)$, $\mathcal{P}(r)$:

$$\mathcal{P}(\alpha_\mu|r) = \begin{cases} p(\alpha_\mu)\,, & \mu \leq K(r) \\ \delta(\alpha_\mu)\,, & \mu > K(r) \end{cases}$$

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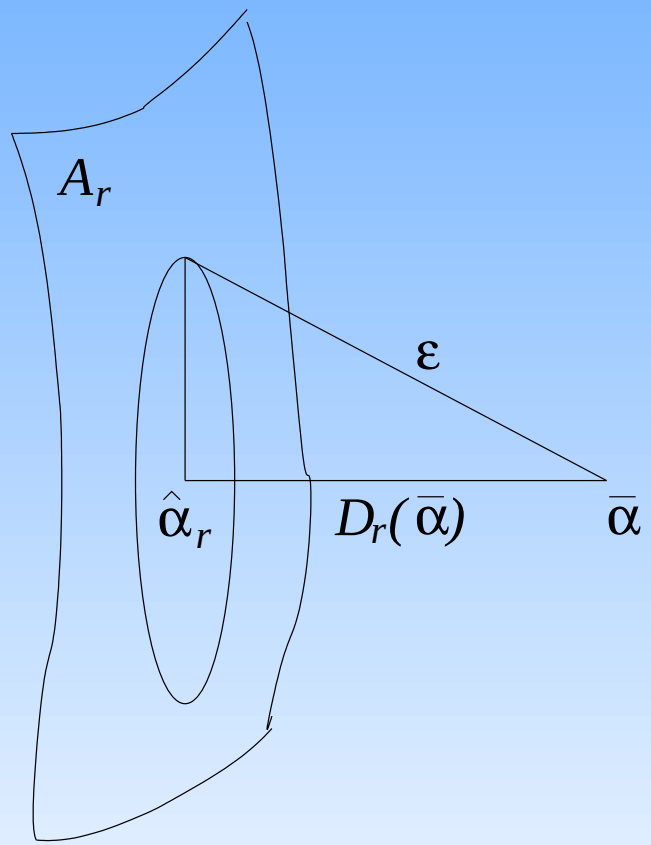

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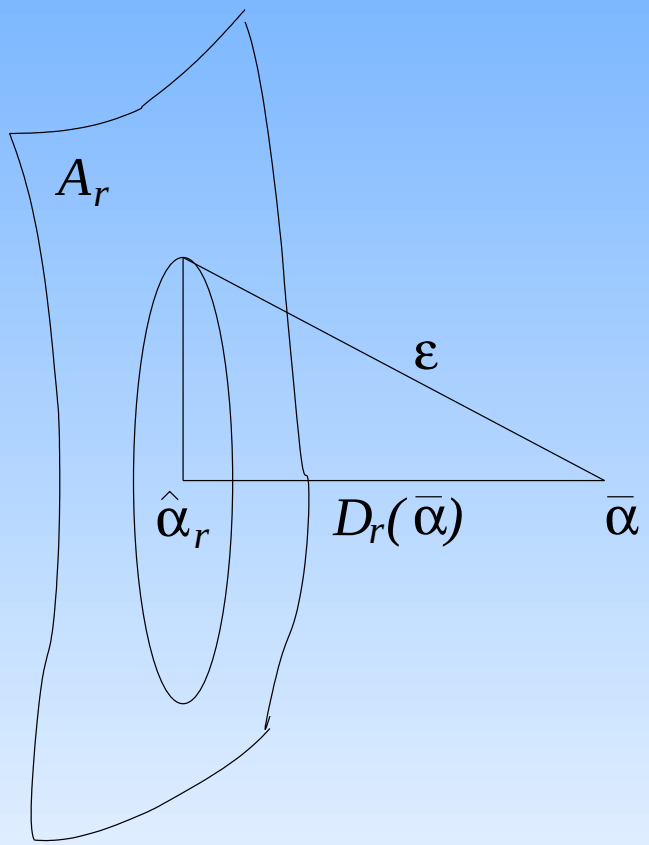

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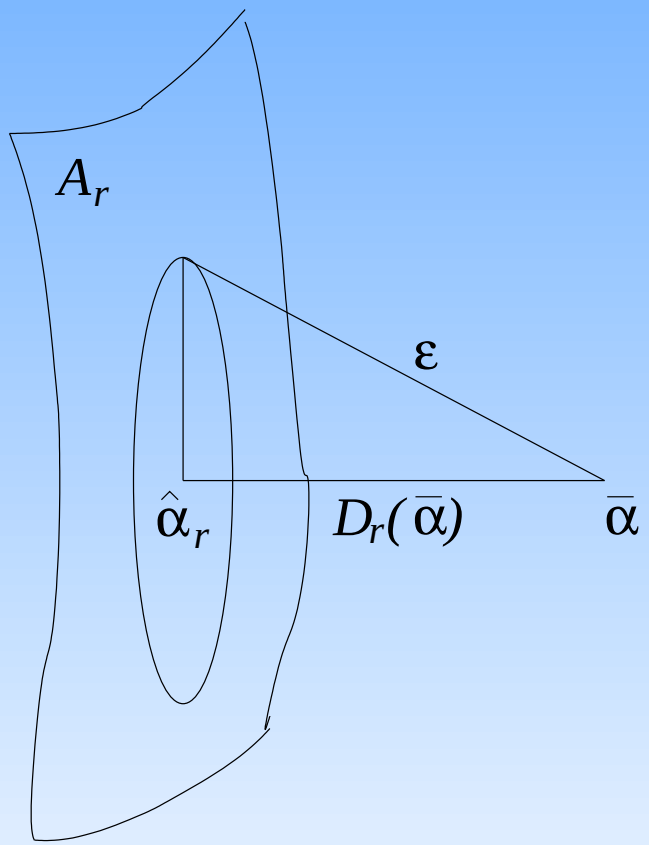

Another complete model!

Density near atypical solutions

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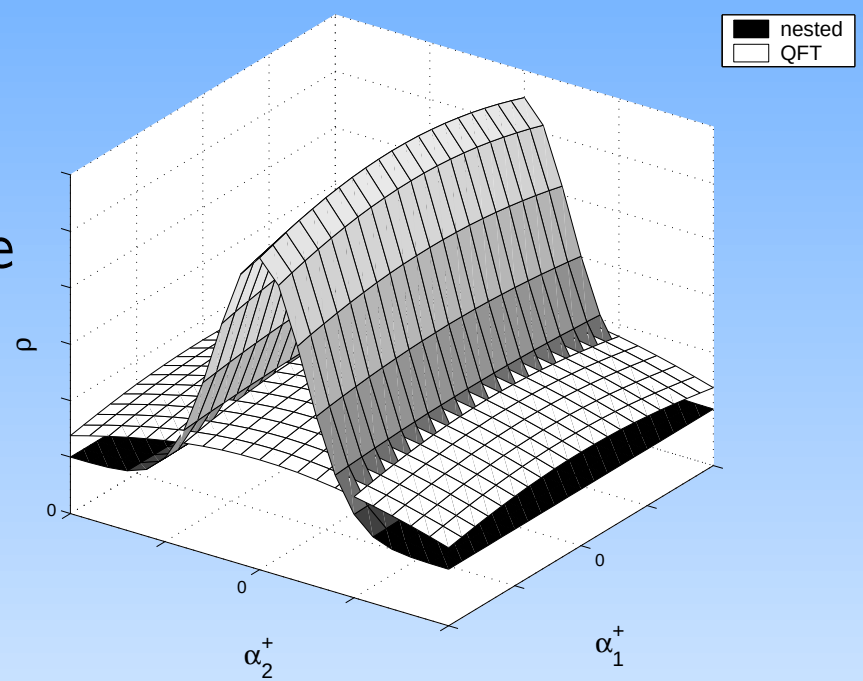

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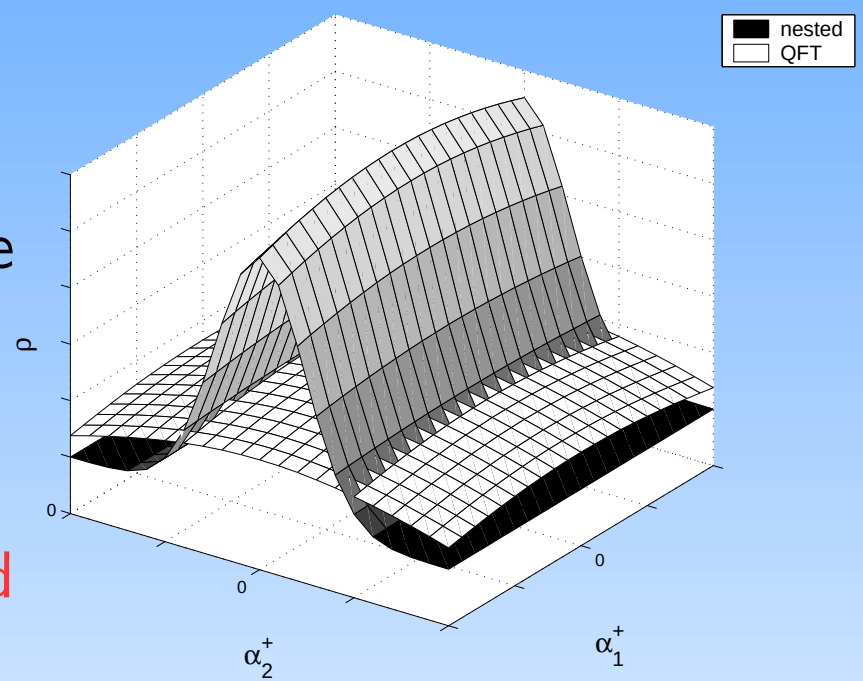

- nested model is at most log worse than the QFT
- QFT may be a power law worse
- for natural (structured) data nested case is better
- alignment may be imperfect for finite precision ϵ

Which model is being used?

- for QFT or nested asymptotics kicks in fast
- asymptotic decay rate should signify the model

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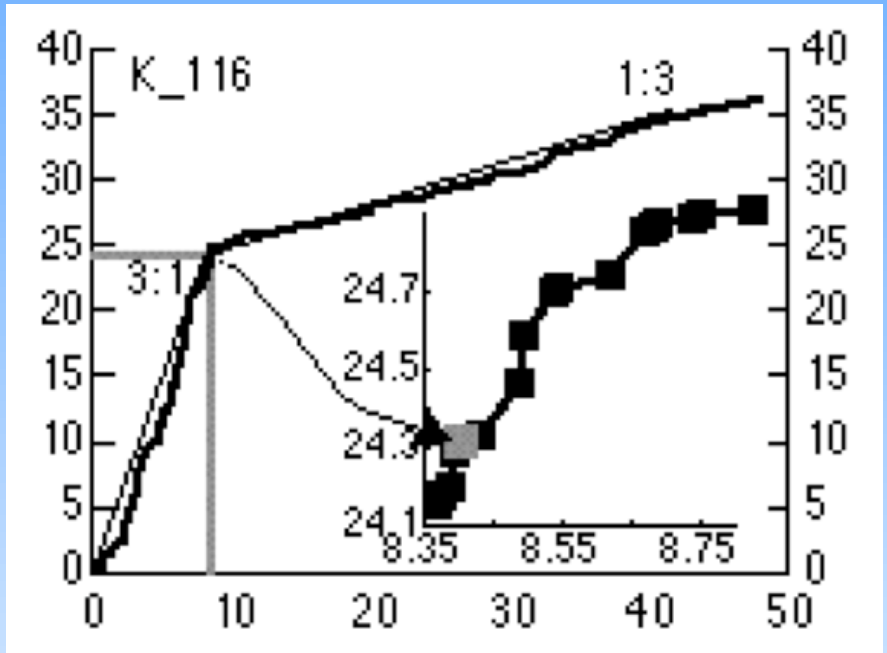

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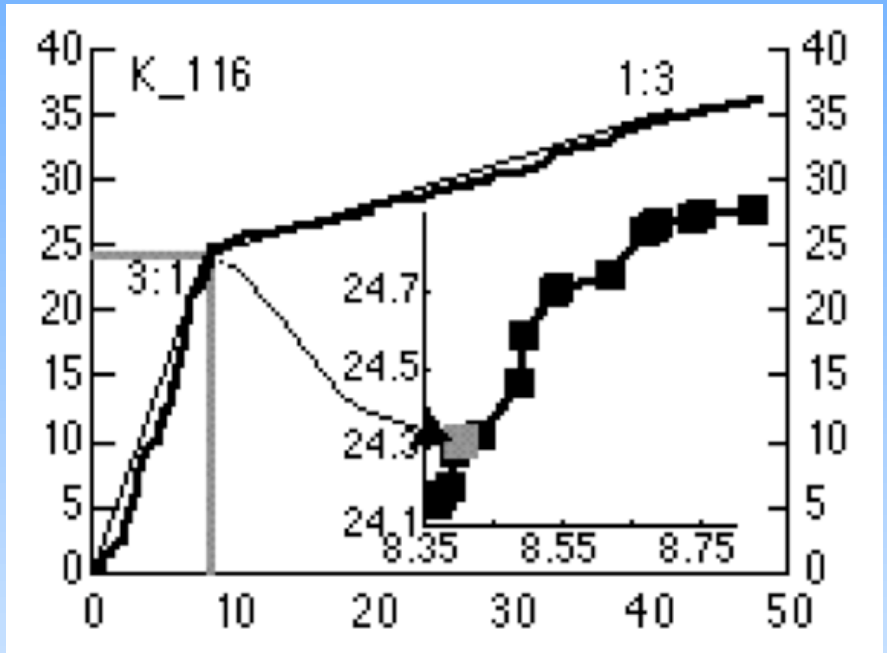

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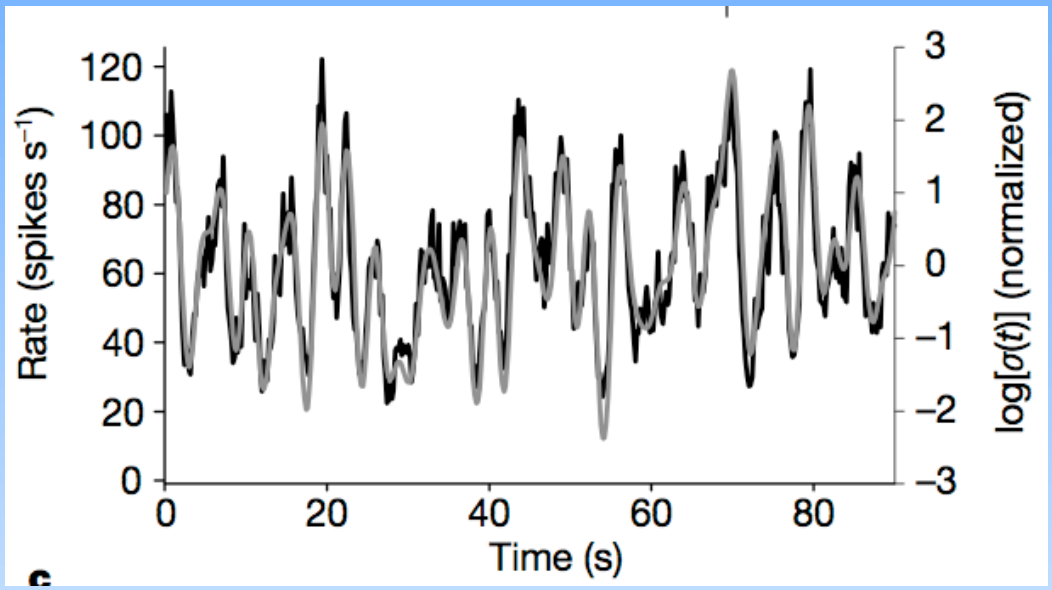
- maybe FDT? $\frac{\partial \Lambda}{\partial N} = -\zeta_N \Lambda^\nu$

Fluctuations (drifting target) and dissipation (learning curve)



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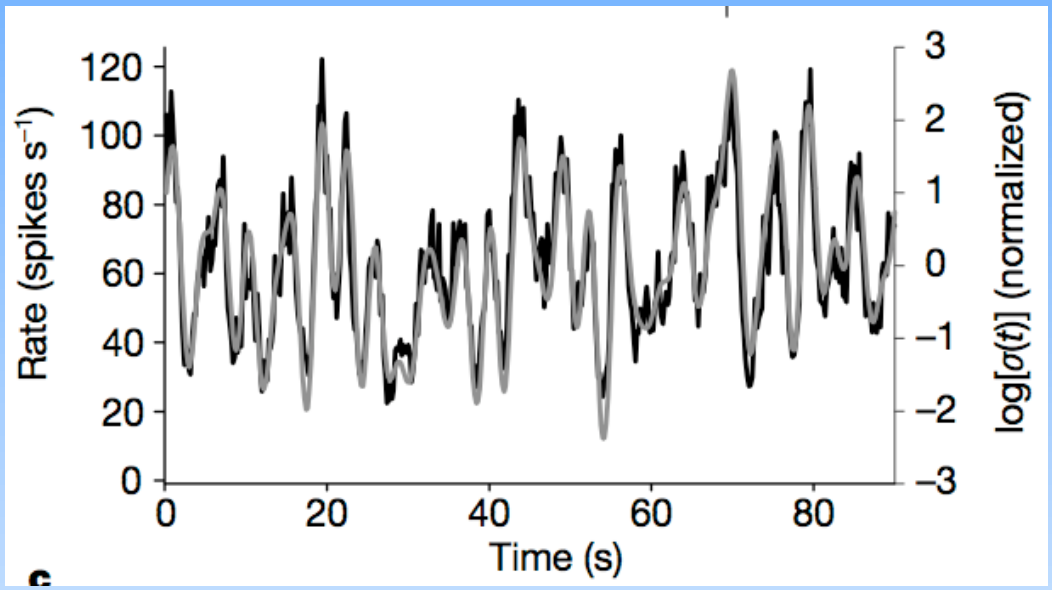

$$\Delta_{\mathrm{rms}} = \left\{ \nu^{1/\nu} \frac{\Gamma\left(\frac{3}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)} \right\}^{1/2} \left(\frac{\Omega}{\zeta}\right)^{1/(2\nu)}$$

The hidden extras. . .

Which complexity do we want to define?

- complexity of dynamics that generates a time series (not computational or descriptive complexity); thus it must be zero for totally random and for easily predictable processes
- usable for Occam–style punishment in statistical inference
- expressible in conventional physical terms
- must be attached to an ensemble, not a single realization

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The divergent subextensive term measures complexity uniquely!

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If sufficient statistics exist, then $C_K \approx I_{\text{pred}}$. Otherwise $C_K > I_{\text{pred}}$.

C_K is unique up to a constant.

Not finite size scaling!

What's next?

extraction separating predictive information from non–predictive using the 'relevant information' technique

physics of phase transitions, connection to subextensive statistical mechanics

statistics extensions of MDL (predictive information *is* a property of the data, not of the model)

learning unification of approaches: Bayesian, SRM, MDL, Cucker-Smale. . .

bioinformatics what is predictive information of natural symbolic sequences? (DNA, languages, spike trains) can we use changes in predictability for data partitioning? for model building?

dynamical systems theory what is predictive information and complexity of various systems?