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On the influence of the gravitating vacuum on the dynamics of homogeneous isotropic models in gauge theories of gravity

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Abstract. The analysis of solutions of gravitational equations for homogeneous isotropic models in the presence of a gravitating vacuum (cosmological constant χ) is given in the frame of various gauge theories of gravity on the basis of qualitative theory of dynamic systems. The equation of state of matter is given in the form of a linear dependence of the pressure p on the energy density ρ with restriction $p > \rho/3$. It is shown that regularization of metric derivatives takes place in the case of sufficiently high values of χ, and some are obtained which are regular in the metrics solutions for superdense gravitating systems.

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1. Introduction

One of the most important consequences of gauge theories of gravity (Poincaré gauge theory (PGT), metric–affine gauge theory (MAGT) etc) is the conclusion about the important regularizing role of a gravitating vacuum†. In particular, this conclusion is displayed in the existence of the vacuum gravitational repulsion effect (VGRE) [1, 2]. This effect can take place when the energy density of a gravitating vacuum (cosmological constant χ) is sufficiently high and can play an important role during the early steps of cosmological expansion in inflationary models and also in the case of gravitating systems at extreme conditions (extremely high energy densities, pressures, temperatures). For the first time VGRE was investigated in the case of homogeneous isotropic models in the frame of PGT based on a sufficiently general gravitational Lagrangian L_G including both a scalar curvature and terms quadratic in the curvature and torsion tensors [1]. As was shown in [1], VGRE follows directly from the generalized cosmological Friedmann equation (GCFE) for the scale factor in the Robertson–Walker metrics [4, 3]:

$$\frac{k}{R^2} + \left\{\frac{\mathrm{d}}{dt}\ln\left[R\sqrt{|1-\beta(\rho+4\chi-3p)|}\right]\right\}^2 = \frac{1}{6f_0}\frac{\rho+\chi-\frac{\beta}{4}(\rho+4\chi-3p)^2}{1-\beta(\rho+4\chi-3p)} \qquad (k=0,\pm 1) \tag{1}$$

† We consider a gravitating vacuum as a system with the energy–momentum tensor equal to $\chi g_{\mu\nu}$ (χ = constant, $g_{\mu\nu}$ is the metric tensor). The other way of introducing a gravitating vacuum in the theory is in connection with the ordinary cosmological constant χ in the gravitational equations.

where ρ is the energy density, p is the pressure, $f_0 = 1/16\pi G$ (G is Newton's gravitational constant), $\beta = -f/3f_0^2 > 0$, f is a certain combination of coefficients f_i quadratic in the curvature terms of L_G. VGRE appears evidently in the case of systems including radiation ($p_r = \rho_r/3$) in the presence of a gravitating vacuum ($\rho_v = \chi > 0,\ p_v = -\rho_v$). In this case (1) has the form of an ordinary cosmological Friedmann equation in general relativity theory (GR) with effective gravitational constant $\tilde{G} = G(1-4\beta\chi)^{-1}$. If $\chi > 1/4\beta$ we have $\tilde{G} < 0$ which leads to the gravitational repulsion effect. Taking into account VGRE, regular solutions for superdense gravitating systems having essentially non-Einsteinian properties were discussed in [2]. These systems possess a lower limit for admissible values of energy densities in the region of extremely high densities $\rho \sim \beta^{-1}$.

VGRE also takes place in MAGT in the case $\beta > 0$, because GCFE is also valid in MAGT [3].

As the behaviour of solutions of (1) depends essentially on an equation of state of matter, we need to know the behaviour of this equation at extreme conditions to investigate the influence of a gravitating vacuum on the dynamics of gravitating systems. Since we do not know this equation it is certainly of interest to investigate various models with phenomenologically given equations of state. So the influence of a gravitating vacuum on the dynamics of homogeneous isotropic models was discussed in [5] in the case of an equation of state

$$p = \frac{1-\gamma}{3}\rho \qquad \gamma = \text{constant} \tag{2}$$

where the parameter γ satisfies the following restriction: $0 < \gamma \leqslant 1$. The analysis of various solutions with all possible values of χ was given in [5] on the basis of qualitative theory. The conclusion about the important regularizing role of a gravitating vacuum in the case $1/4\beta < \chi < 1/\gamma(4-\gamma)\beta$ was obtained. However, the restriction $p < \rho/3$ used in [5] may not take place if classical scalar fields make a great contribution to the energy density of matter. As the relation $p_s = \rho_s$ is valid for massless scalar fields in the isotropic case the following condition for matter: $p > \rho/3$ can take place† [6]. In connection with this the influence of a gravitating vacuum on the dynamics of homogeneous isotropic systems is investigated in this paper in the case $p > \rho/3$. For simplicity we use the equation of state (2) with $\gamma < 0$. Using the qualitative theory for dynamic systems we show that the character of the solutions can vary essentially with varying γ.

2. Equations of dynamic systems on the plane $\{\rho, H\}$

From a geometric point of view homogeneous isotropic models are described in PGT by two functions of time [4]: the scale factor $R(t)$ and torsion function $S(t)$; in MAGT in the limit to the Weyl–Cartan spacetime we also have the non-metricity function $Q(t)$‡. Two curvature functions A and B are determined as follows:

$$A = \frac{\left[\dot{R} - 2R(S - \frac{1}{4}Q)\right]^{\cdot}}{R} \qquad B = \frac{k + \left[\dot{R} - 2R(S - \frac{1}{4}Q)\right]^2}{R^2} \tag{3}$$

where a dot denotes differentiation with respect to time. Gravitational equations of PGT and MAGT lead to the following expressions [3,4]§:

† Note that energy density and the pressure of scalar Higgs fields in the symmetric state can be written as $\rho_s + \rho_v$ and $p_s + p_v$, respectively, for limited time intervals.

‡ The theory investigated in this paper is invariant under space inversion transformations. Theory without this invariance contains the second torsion function (see [7]). In the general case in MAGT there are, in addition, two non-metricity functions.

§ Equations (1) and (3)–(5) are obtained from the corresponding equations of [3,4] by the substitution $\rho \to \rho + \chi$ and $p \to p - \chi$.

$$A = -\frac{1}{12f_0}\frac{\rho + 3p - 2\chi + \frac{1}{2}\beta(\rho + 4\chi - 3p)^2}{1 - \beta(\rho + 4\chi - 3p)} \qquad B = \frac{1}{6f_0}\frac{\rho + \chi - \frac{1}{4}\beta(\rho + 4\chi - 3p)^2}{1 - \beta(\rho + 4\chi - 3p)} \tag{4}$$

$$S - \tfrac{1}{4}Q = -\tfrac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\ln|1 - \beta(\rho + 4\chi - 3p)|\,. \tag{5}$$

In the frame of PGT we are to put $Q = 0$ in (3) and (5), then in the frame of MAGT torsion vanishes ($S = 0$) because of the gravitational equations [3]. Expressions (3) and (4) for the function B lead to GCFE (1). Expressions (3)–(5) for the function A together with the conservation law

$$\dot{\rho} - 3H(\rho + p) = 0 \qquad \left(H = \frac{\dot{R}}{R}\right) \tag{6}$$

lead to the following equation:

$$\begin{aligned}\dot{H}&\left[1 + \frac{3\beta}{2}\frac{\left(1 - 3\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)(\rho + p)}{1 - \beta(\rho + 4\chi - 3p)}\right] + H^2 - \frac{3\beta}{2}H^2\frac{\rho + p}{1 - \beta(\rho + 4\chi - 3p)}\\ &\times\left\{\left(1 - 3\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)\left[2 + 3\frac{\mathrm{d}p}{\mathrm{d}\rho} + \frac{3\beta\left(1 - 3\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)(\rho + p)}{1 - \beta(\rho + 4\chi - 3p)}\right] - 9\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2}(\rho + p)\right\}\\ &= -\frac{1}{12f_0}\frac{\rho + 3p - 2\chi + \frac{1}{2}\beta(\rho + 4\chi - 3p)^2}{1 - \beta(\rho + 4\chi - 3p)}\,.\end{aligned} \tag{7}$$

Equations (6) and (7) determine the behaviour of dynamic systems on the plane $\{\rho, H\}$ if the equation of state $p = p(\rho)$ is given. GCFE is the integral of (7) and equation (6) leads to the following integral:

$$R = \exp\left(-\frac{1}{3}\int\frac{\mathrm{d}\rho}{\rho + p(\rho)}\right)\,. \tag{8}$$

Taking into account (8) the solution of GCFE can be represented in the form [2]

$$\begin{aligned}t_2 - t_1 = \int_{R_1}^{R_2}&\left[1 + \tfrac{1}{2}\beta(\rho - 8\chi + 9p) - \tfrac{9}{2}\beta(\rho + p)\frac{\mathrm{d}p}{\mathrm{d}\rho}\right]\left[1 - \beta(\rho + 4\chi - 3p)\right]^{-1/2}\\ &\times\left\{-k\left[1 - \beta(\rho + 4\chi - 3p)\right] + \frac{1}{6f_0}\left[\rho - \tfrac{1}{4}\beta(\rho + 4\chi - 3p)^2\right]R^2\right\}^{-1/2}\mathrm{d}R\,.\end{aligned} \tag{9}$$

If the equation of state is given in the form (2) with $\gamma < 0$ then we obtain the following relation from (7) and (6) and their integrals (1), (8) and (9) [5]:

$$\begin{aligned}\dot{H} = &\{[1 - \beta(\gamma\rho + 4\chi)][2(1 - 4\beta\chi) + \gamma(2 - \gamma)\beta\rho]\}^{-1}\\ &\times\left\{\frac{1}{6f_0}\left[\tfrac{1}{2}\gamma^2\beta\rho^2 + (2 - \gamma + 4\gamma\beta\chi)\rho - 2\chi(1 - 4\beta\chi)\right][\beta(\gamma\rho + 4\chi) - 1]\right.\\ &\left.+\left[\gamma^2(2 - \gamma)\beta^2\rho^2 + \left(16 - 7\gamma + \gamma^2\right)(1 - 4\beta\chi)\gamma\beta\rho - 2(1 - 4\beta\chi)^2\right]H^2\right\}\end{aligned} \tag{10}$$

$$\dot{\rho} = (\gamma - 4)\rho H \tag{11}$$

$$\frac{k}{R^2}[\beta(\gamma\rho+4\chi)-1]^2+\left[\tfrac{1}{2}\gamma(2-\gamma)\beta\rho+1-4\beta\chi\right]^2H^2$$
$$=\frac{1}{6f_0}\left[\tfrac{1}{4}\gamma^2\beta\rho^2+(2\gamma\beta\chi-1)\rho-\chi(1-4\beta\chi)\right][\beta(\gamma\rho+4\chi)-1] \tag{12}$$

$$R=c_\gamma\rho^{1/(\gamma-4)} \tag{13}$$

$$t_2-t_1=\int_{\rho_1}^{\rho_2}\frac{1}{(4-\gamma)\rho}\left[\tfrac{1}{2}\gamma(2-\gamma)\beta\rho+1-4\beta\chi\right][\beta(\gamma\rho+4\chi)-1]^{1/2}$$
$$\times\left\{\frac{1}{6f_0}\left[\tfrac{1}{4}\gamma^2\beta\rho^2+(2\gamma\beta\chi-1)\rho-\chi(1-4\beta\chi)\right]\right.$$
$$\left.-\frac{k}{c_\gamma^2}\rho^{2/(4-\gamma)}[\beta(\gamma\rho+4\chi)-1]\right\}^{-1/2}\mathrm{d}\rho \tag{14}$$

where c_γ is the integration constant. Equation (5) has the following form:

$$S-\tfrac{1}{4}Q=\frac{\beta\gamma\dot{\rho}}{4(1-\beta(\gamma\rho+4\chi))}. \tag{15}$$

3. Qualitative analysis of solutions

Let us analyse solutions given by (14) for the dependence on the value of χ on the basis of the qualitative theory of dynamic systems [8] (as in [5]). The conditions $\dot{H}=\dot{\rho}=0$ give from (10) and (11) the following particular points in physical region ($\rho\geqslant 0$, H is the real value):

(i) points K_1 and K_2 ($\rho=0, H=\pm\sqrt{\chi/6f_0}$) in the case $\chi>0$ and $\chi\neq 1/4\beta$. These points are stable and unstable knots, respectively, and correspond to the de Sitter solution of GR. If $\chi=0$ then K_1 and K_2 unite and form a point of complicated equilibrium;

(ii) points L_1 and L_2 ($\rho=0, H=\pm\sqrt{1/3f_0\gamma(\gamma-2)}$) if $\chi=1/4\beta$. They correspond to a complicated equilibrium state and have a structure of stable and unstable knots, respectively;

(iii) saddle-type point M ($\rho=(1/\gamma^2\beta)\{\gamma-2-4\gamma\beta\chi+[(2-\gamma+4\gamma\beta\chi)^2--4\gamma^2\beta\chi(4\beta\chi-1)]^{1/2}\}$, $H=0$) in the case $0<\chi<1/4\beta$.

Note that because of (10) the derivative $\mathrm{d}H/\mathrm{d}t$ diverges at the point

$$\rho=\rho_\infty=\frac{2(4\beta\chi-1)}{\gamma(2-\gamma)\beta}$$

which is in the physical region when $\chi<1/4\beta$.

Putting $H=0$ in (12) we find values $\rho=\rho_i$ ($i=1,2,3$) limiting the region of varying of ρ for phase trajectories. To do this we write (12) in the following form [5]:

$$\frac{6(2-\gamma)^2f_0}{\gamma}(\rho-\rho_0)^2H^2=(\rho-\rho_1)(\rho-\rho_2)(\rho-\rho_3)$$
$$=(\rho-\rho_1)\left((\rho-\rho_2^{(0)})(\rho-\rho_3^{(0)})-\frac{24kf_0}{\gamma c_\gamma^2}\rho^{2/(4-\gamma)}(\rho-\rho_1)\right) \tag{16}$$

where

$$\rho_0=\frac{2(4\chi\beta-1)}{\gamma(2-\gamma)\beta}=\rho_\infty \qquad \rho_2^{(0)}=\frac{2}{\gamma^2\beta}(1-2\beta\gamma\chi+\sqrt{1-\gamma(4-\gamma)\beta\chi})$$
$$\rho_1=\frac{1-4\beta\chi}{\gamma\beta} \qquad \rho_3^{(0)}=\frac{2}{\gamma^2\beta}(1-2\beta\gamma\chi-\sqrt{1-\gamma(4-\gamma)\beta\chi}).$$

Considering positive values of χ ($\chi > 0$) note that the point $\rho_2^{(0)}$ is always in the physical region, ρ_0 is in the physical region if $\chi < 1/4\beta$ and points ρ_1 and $\rho_3^{(0)}$—in the case $\chi > 1/4\beta$. The value of ρ_1 is the same for any $k = 0, \pm 1$, but values of ρ_2 and ρ_3 depend on the type of model: ρ_2 and ρ_3 coincide with $\rho_2^{(0)}$ and $\rho_3^{(0)}$ in the case of a flat model ($k = 0$) and are displaced to the corresponding directions in the case of open ($k = -1$) and closed ($k = 1$) models.

In figures 1–4 one can see phase trajectories corresponding to various solutions of investigated dynamic system. Full curves correspond to flat models, chain curves—to closed models and dotted curves—to open models.

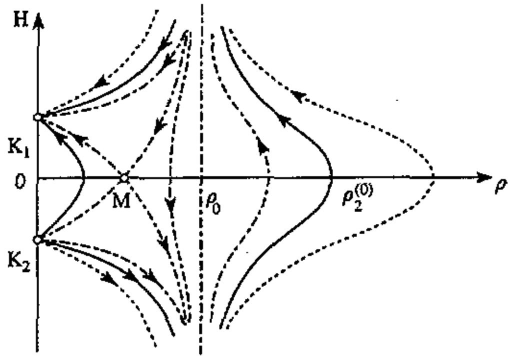

Figure 1.

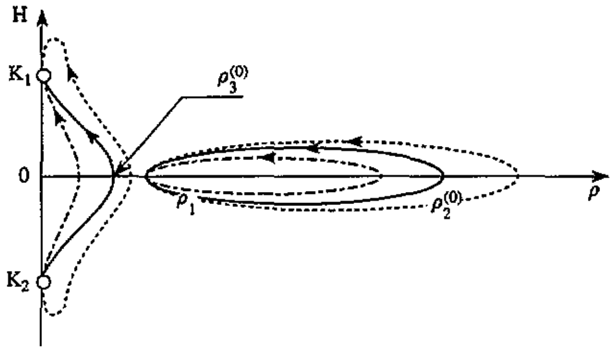

Figure 2.

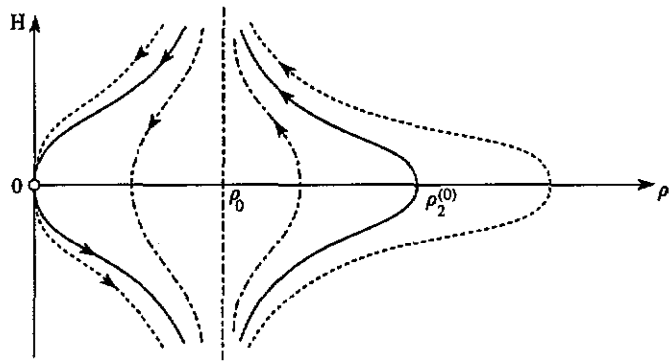

Figure 3.

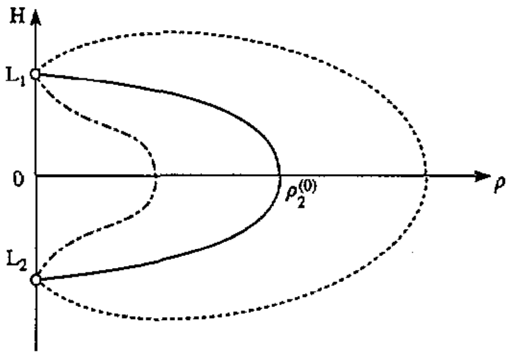

Figure 4.

(i) $0 < \chi < 1/4\beta$ (figure 1). All solutions for flat and open models and some solutions for close models are singular in the metrics derivatives (H and $\dot{H}$) at the point ρ_0, but the metrics itself is a regular function. Close models are regular in the metrics, its derivatives and torsion (non-metricity) solutions for some values of c_γ analogously to corresponding solutions in GR. These solutions are described by a branch to the left from M in figure 1. There are some c_γ for which solutions singular in H and $\dot{H}$ for close models do not exist. Regular transition from compression to expansion takes place in the point ρ_2. In this point $R = R_{\min} > 0$ and $\mathrm{d}R/\mathrm{d}t = 0$. Torsion (non-metricity) determined according to (15) vanishes in ρ_2, K_1, K_2, M.

(ii) $\chi > 1/4\beta$ (figure 2). All points ρ_1, ρ_2 and ρ_3 are in physical region and ρ_3 $(k = 1) < \rho_3^{(0)} < \rho_3$ $(k = -1) < \rho_1 < \rho_2$ $(k = 1) < \rho_2^{(0)} < \rho_2$ $(k = -1)$. Phase trajectories represent solutions of two types. The first type corresponds to solutions with the de Sitter asymptotics (points K_1 and K_2) and describes smooth transition from compression to expansion. These solutions are regular in the metrics as well as in the torsion (non-metricity). The regularizing influence of the gravitating vacuum is displayed in the prevention of the metrics derivatives' singularities that take place for the corresponding solution in figure 1. The second type of solutions corresponds to superdense systems (closed curves in figure 2). The dynamics of these systems has the character of oscillation between minimum ρ_1 and maximum ρ_2 values of the energy density. These solutions are regular in the metrics but in ρ_1 torsion (non-metricity) diverges. Note that torsion (non-metricity) vanishes at the points ρ_2, ρ_3, K_1 and K_2.

(iii) $\chi = 0$ (figure 3). The picture of phase trajectories in this case can be obtained from figure 1 by tightening of three particular points K_1, K_2, M to point $O(\rho = 0, H = 0)$. The point O is equilibrium state for the system and it is stable if $H > 0$ and unstable if $H < 0$.

(iv) $\chi = 1/4\beta$ (figure 4). Points L_1, L_2 and ρ_2 $(k = 1) < \rho_2^{(0)} < \rho_2$ $(k = -1)$ are in a physical region. Solutions are regular in the metrics and in the torsion (non-metricity) defined in this case by formula $S - 1/4Q = (4 - \gamma)H/4$. At the point ρ_2 torsion (non-metricity) is equal to zero, but at points L_1 and L_2 torsion (non-metricity) is non-zero though at these points $\rho = 0$. When $\rho \to 0$ (points L_1 and L_2) the scale factor $R(t)$ grows exponentially like the de Sitter solutions:

$$R = R_0 \exp\left(\pm\sqrt{\frac{b}{a}}t\right) \qquad k = 0$$

$$R = \sqrt{\frac{1}{ab}} \operatorname{sh}\left(\pm\sqrt{\frac{b}{a}}t + C_1\right) \qquad k = -1$$

$$R = \sqrt{\frac{1}{ab}} \operatorname{ch}\left(\pm\sqrt{\frac{b}{a}}t + C_1\right) \qquad k = 1$$

where $a = (\gamma - 2)/2, b = (6f_0\beta\gamma)^{-1}$; R_0, C_1 and C_2 are integration constants. Note that $H(\rho = 0) = \pm\sqrt{b/a}$.

Solutions (14) corresponding to negative values of χ ($\chi < 0$) are similar to solutions discussed in the case $\chi = 0$. In the case $1/\beta\gamma(4-\gamma) < \chi < 0$ solutions for open and flat models with any values of c_γ and solutions for closed models with some values of c_γ exist. In the case $\chi \leqslant 1/\beta\gamma(4-\gamma)$ there are only solutions for open models.

4. Conclusion

The qualitative analysis carried out above for homogeneous isotropic models with the equation of state (2) with $p > \rho/3$ confirms the conclusion about regularizing the role of a gravitating vacuum with sufficiently high energy density ($\chi > 1/4\beta$) which was obtained for the first time in [1]. Note that the type of regularization depends essentially on the equation of state of the matter. While in the case of radiation ($p = \rho/3$) we have the regularization of the metrics because of VGRE [1] and in the case of an equation of state in the form (2) with $p < \rho/3$ torsion is regularized [5], in the discussed case ($p > \rho/3$) a gravitating vacuum leads to regularization of the metrics' derivatives. Regular solutions with the de Sitter asymptotics ($\chi > 1/4\beta$) can be used for the construction of regular inflationary models [9]. Solutions obtained for superdense gravitating systems without phase transitions 'vacuum $\leftrightarrow$ matter' (ρ_v = constant) are also of certain interest. Taking into account the evolution of the equation of state of matter the torsion singularity at the point with minimum energy density ($\rho = \rho_1$) can probably be avoided. This is connected with the fact that in the case of the equation of state (2) with $p < \rho/3$ a minimum value of the energy density for superdense systems occurs at the point ρ_2 where torsion vanishes [5]. Equally, taking into consideration the evolution of the equation of state we can try to avoid singularities of the metrics' derivatives in solutions with $\chi < 1/4\beta$ because corresponding solutions in the case $p < \rho/3$ are regular in the metrics and its derivatives [5]. However, these problems are to be specially investigated.

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