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THEORETICAL AND MATHEMATICAL PHYSICS

On the Interaction of Point Charges in an Arbitrary Domain*

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Abstract—We develop a systematic approach to calculating the electrostatic force between point charges in an arbitrary domain with arbitrary boundary conditions. When the boundary is present, the simple expression for the force acting on a charge as "the charge times the field it is placed in" becomes ill-defined. However, this rule can be salvaged if the field in question is redefined to only include all terms that do not diverge at the charge position, in particular, those due to the charge itself. The proof requires handling the self-action energy divergence for point charges, which is accomplished by means of a geometrical regularization. © *2003 MAIK "Nauka/Interperiodica".*

I. INTRODUCTION

It is trivial to determine the force exerted by an external field[1] on a point charge in an otherwise empty space: by definition, "the force is equal to the charge times the field it is placed in." In particular, if the field in question is created by some other point charges, this rule, known by many from high school, still holds.

However, the situation changes drastically when a set of point charges creates the field inside an arbitrary domain with a boundary of some physical origin (reflected in the appropriate boundary conditions). Now the very notion of "the field the charge is placed in" becomes ill-defined. For example, a naive treatment of a single-charge problem might lead one to the entirely wrong conclusion that, since the entire field in the problem is due to the charge itself (there are no other sources!), "the field it is placed in" is zero, so there is no force at all.

A slightly more sophisticated physicist would argue that only that part of the field which diverges as $1/r^2$ near the charge is really created by it, while the rest is due to the boundary conditions, which represent mathematically the rearrangement of *other* physical charges at the boundary. Therefore, it is precisely what remains after subtracting the singular part that now gives "the field the charge is placed in." Unfortunately, such treatment leaves one in a somewhat awkward position of, first, calculating potentials and fields rigorously and, then, lowering the plank and using hand-waving arguments to derive forces from them. It is also not clear whether the conjecture about which part of the total field contributes to the force is always valid.

Thus, it seems appealing to show that the physical arguments can be backed by an accurate mathematical proof demonstrating that the *adjusted* rule, "the force is equal to the charge times the part of the field that does not diverge at the charge's location," is either universal or limited by certain conditions. To do this, one should turn to the most fundamental energy conservation argument which gives the force as the negative gradient of the energy in the charge's position. This approach also does not turn out to be straightforward, since the energy is infinite in the presence of point charges due to their self-action.

Perhaps because of these difficulties, as well as the misleading apparent simplicity of the problem, our literature search, which encompassed, in particular, [1–10] and many other books on the subject, revealed no ready result (except in [6], which we discuss in Section 4). So we give a careful derivation of the general expression for the force on point charges in this paper. It consists of the regularization of the problem, calculation of the force from the (regularized and finite) energy, and then taking the singular limit. The result agrees with one's intuitive expectations.

II. ELECTROSTATICS PROBLEM WITH VOLUME POINT CHARGES: POTENTIAL AND ENERGY

Consider an arbitrary three-dimensional domain D with a perfectly conducting boundary S and some N point electrical charges inside. The electrical potential $\psi(\mathbf{r})$, in this case, is determined by the following Dirichlet boundary value problem (we use SI units throughout the paper):

$$\Delta\psi = -\frac{1}{\varepsilon_0}\sum_{i=1}^{N} q_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \mathbf{r}, \mathbf{r}_i \in D; \tag{1}$$

$$\psi|_S = 0. \tag{2}$$

[1] In fact, the word "external" means that the field is produced by some independent sources, and the boundaries, if any, are far away from the charge.

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Here, $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ is the vector radius of a point, and $\mathbf{r}_i = x_i\mathbf{e}_x + y_i\mathbf{e}_y + z_i\mathbf{e}_z$ specifies the ith charge position, with $\mathbf{e}_\alpha$, $\alpha = x, y, z$, being the unit vectors in the direction of the corresponding Cartesian axes.

By the superposition principle, the potential $\psi(\mathbf{r})$ is merely the sum of the potentials induced by each charge separately,

$$\psi(\mathbf{r}) = \kappa\sum_{j=1}^{N} q_j G(\mathbf{r}, \mathbf{r}_j), \tag{3}$$

$$\equiv \kappa\sum_{j=1}^{N} q_j\left[\frac{1}{|\mathbf{r}-\mathbf{r}_j|} + G_R(\mathbf{r}, \mathbf{r}_j)\right], \tag{4}$$

where $\kappa = 1/4\pi\varepsilon_0$, and $G_R(\mathbf{r}, \mathbf{r}_j)$ is the regular part of the Green's function $G(\mathbf{r}, \mathbf{r}_j)$ of the corresponding boundary value problem [set $q_j = 1$, $q_i = 0$, $i \neq j$ in Eq. (1)]. Both functions are, of course, symmetric in their arguments,

$$G(\mathbf{r}, \mathbf{r}_j) = G(\mathbf{r}_j, \mathbf{r}), \quad G_R(\mathbf{r}, \mathbf{r}_j) = G_R(\mathbf{r}_j, \mathbf{r}). \tag{5}$$

Furthermore, we can rewrite Eq. (4), splitting the potential in a sum of its singular and regular parts,

$$\psi(\mathbf{r}) = \kappa\sum_{j=1}^{N}\frac{q_j}{|\mathbf{r}-\mathbf{r}_j|} + \psi_R(\mathbf{r}), \tag{6}$$

$$\psi_R(\mathbf{r}) \equiv \kappa\sum_{j=1}^{N} q_j G_R(\mathbf{r}, \mathbf{r}_j), \tag{7}$$

where $\psi_R(\mathbf{r})$ is a regular function satisfying the Laplace equation everywhere in D [by continuity, this holds also at any regular point[2] of the boundary S, although this is irrelevant to our discussion]. Note that both the potential ψ and its regular part ψ_R actually depend on the positions of the charges $\mathbf{r}_i$, as well as on the observation point $\mathbf{r}$, which is reflected in the full notation

$$\psi(\mathbf{r}) \equiv \psi(\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N), \tag{8}$$

$$\psi_R(\mathbf{r}) \equiv \psi_R(\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N). \tag{9}$$

We assume that the potential is known, and we are interested in finding the force $\mathbf{F}^i$ acting on the charge q_i. From the energy conservation for the considered problem, the force is given by (cf. [6])

$$\mathbf{F}^i = -\frac{\partial}{\partial\mathbf{r}_i}W_D, \quad \frac{\partial}{\partial\mathbf{r}_i} = \frac{\partial}{\partial x_i}\mathbf{e}_x + \frac{\partial}{\partial y_i}\mathbf{e}_y + \frac{\partial}{\partial z_i}\mathbf{e}_z, \tag{10}$$

where W_D is the energy of the field in the volume D,

$$W_D = \frac{\varepsilon_0}{2}\int_D (\nabla\psi)^2 dV. \tag{11}$$

[2] We allow for boundary singularities, such as sharp edges and spikes, provided that the Meixner-type finite energy condition [11] is satisfied near them; in particular, the domain D can be infinite.

Note that we alternatively write ∇ or $\partial/\partial\mathbf{r}$ for the gradient, whatever seems proper in a particular expression.

The problem is, however, that the above integral obviously diverges due to self-interaction of the point charges (the energy of a single point charge is infinite). We are going to show that even though the energy for a given point charge distribution is infinite, the *difference* between its two values corresponding to any two charge configurations *is finite* (for an arbitrary boundary shape) and goes to zero when one charge distribution tends to the other. Hence, the *force is also finite*, in accordance with common intuition. The situation with energy here is similar to the one arising in the calculation of the Casimir effect [12], which also requires some generalization.

III. REGULARIZED ENERGY AND THE FORCE ON THE CHARGES

We surround each volume charge q_i by a small sphere S_i^ϵ of radius ϵ; we write D_i^ϵ for the ball inside it. We define D^ϵ as D without all domains D_i^ϵ, and S^ϵ as a union of S and all spherical surfaces S_i^ϵ (see Fig. 1). In effect, S^ϵ is the boundary of the domain D^ϵ and $D^\epsilon \longrightarrow D$, $S^\epsilon \longrightarrow S$ when $\epsilon \longrightarrow 0$.

Using Eq. (10), we may now define the force acting on the charge q_i as

$$\mathbf{F}^i = \lim_{\epsilon\to 0}\mathbf{F}^i_\epsilon = -\lim_{\epsilon\to 0}\frac{\partial}{\partial\mathbf{r}_i}W_D^\epsilon, \tag{12}$$

where W_D^ϵ is the regularized energy, that is, the energy of the field in D^ϵ, which is finite. It is important to note the order of operations in Eq. (12): first, take the gradient of the regularized energy in the charge position, then, take the (singular) limit. In principle, we also have to show that the final result does not depend on the regularization chosen, but this task is not easy. We will return to this problem briefly later in this paper.

In view of Eq. (11) and the fact that the total potential given by Eq. (3) or Eq. (6) is regular in D^ϵ, the reg-

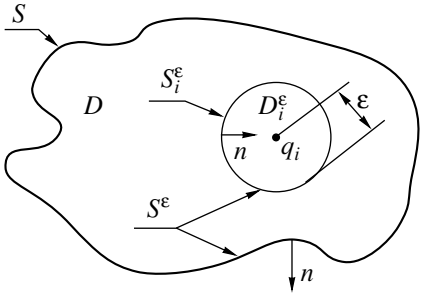

Fig. 1. Volumes, surfaces, and normal directions involved.

ularized energy is

$$W_D^\epsilon \equiv \frac{\varepsilon_0}{2}\int_{D^\epsilon}(\nabla\psi)^2 dV \tag{13}$$

$$= \frac{\varepsilon_0}{2}\int_{S^\epsilon}\psi\frac{\partial\psi}{\partial n}dA - \frac{\varepsilon_0}{2}\int_{D^\epsilon}\psi\Delta\psi dV, \tag{14}$$

where n is the direction of the *outward* normal to S^ϵ (and thus the *inward* normal to the spheres S_i^ϵ). For an infinite domain D, it is assumed here that the potential and its gradient drop at infinity fast enough to make the contribution of integrating over the sphere of a large radius vanishing in the limit, an assumption which has to be verified in each particular case.

Since ψ is harmonic everywhere in D^ϵ, the volume integral on the right of the previous equality vanishes; the remaining surface integral is represented as

$$W_D^\epsilon = \frac{\varepsilon_0}{2}\sum_{k=1}^{N}\int_{S_k^\epsilon}\psi\frac{\partial\psi}{\partial n}dA + \frac{\varepsilon_0}{2}\int_{S}\psi\frac{\partial\psi}{\partial n}dA \tag{15}$$

and then, because of the boundary condition, Eq. (2), as

$$W_D^\epsilon = \frac{\varepsilon_0}{2}\sum_{k=1}^{N}\int_{S_k^\epsilon}\psi\frac{\partial\psi}{\partial n}dA. \tag{16}$$

We are ultimately interested in the limit $\epsilon \longrightarrow 0$, so we need to calculate only the quantities which do not vanish in this limit. The area of integration in each term of the above sum is $O(\epsilon^2)$; therefore, we need to keep track of the integrands that grow at least quadratically in ϵ^{-1}. Bearing this in mind and using Eq. (6) for the potential, we can write the surface integral in Eq. (16) as

$$\int_{S_k^\epsilon}\psi\frac{\partial\psi}{\partial n}dA = \int_{S_k^\epsilon}\frac{\kappa q_k}{|\mathbf{r}-\mathbf{r}_k|}\frac{\partial}{\partial n}\left(\frac{\kappa q_k}{|\mathbf{r}-\mathbf{r}_k|}\right)dA + \int_{S_k^\epsilon}\left[\psi_R(\mathbf{r}) + \sum_{j=1,\,j\neq k}^{N}\frac{\kappa q_j}{|\mathbf{r}-\mathbf{r}_j|}\right]\frac{\partial}{\partial n}\left(\frac{\kappa q_k}{|\mathbf{r}-\mathbf{r}_k|}\right)dA + O(\epsilon). \tag{17}$$

The first term in the above expression is, in fact, the regularized self-energy of the kth charge, $W_{k,\,\mathrm{self}}^\epsilon$. Performing elementary integration, we immediately find that

$$W_{k,\,\mathrm{self}}^\epsilon = \frac{\varepsilon_0}{2}\kappa^2\frac{4\pi q_k^2}{\epsilon} = \frac{\kappa q_k^2}{2\epsilon}. \tag{18}$$

The only feature of the regularized self-energy given by Eq. (18) that is important for our derivation is that *it does not depend on the position of the charge q_k, i.e., on the vector radius $\mathbf{r}_k$.*

The second term of the right-hand side of Eq. (17) can also be simplified if one notices that both ψ_R and $1/|\mathbf{r}-\mathbf{r}_j|$, $j \neq k$, are regular on S_k^ϵ and in D_k^ϵ. Therefore, their change within the small surface S_k^ϵ is of order ϵ. Thus, Eq. (17) may be rewritten as

$$\frac{\varepsilon_0}{2}\int_{S_k^\epsilon}\psi\frac{\partial\psi}{\partial n}dA = W_{k,\,\mathrm{self}}^\epsilon + \frac{\varepsilon_0}{2}\left[\psi_R(\mathbf{r}_k) + \sum_{j=1,\,j\neq k}^{N}\frac{\kappa q_j}{|\mathbf{r}_k-\mathbf{r}_j|}\right] \times \int_{S_k^\epsilon}\frac{\partial}{\partial n}\left(\frac{\kappa q_k}{|\mathbf{r}-\mathbf{r}_k|}\right)dA + O(\epsilon) \tag{19}$$

$$= W_{k,\,\mathrm{self}}^\epsilon + \frac{q_k}{2}\left[\psi_R(\mathbf{r}_k) + \sum_{j=1,\,j\neq k}^{N}\frac{\kappa q_j}{|\mathbf{r}_k-\mathbf{r}_j|}\right] + O(\epsilon),$$

and the integration here yielding the factor 4π is again elementary. This asymptotic equality may be differentiated in $\mathbf{r}_i$ with the same estimate of the remaining term.

Introducing now the previous expression into Eq. (16), we obtain

$$W_D^\epsilon = \sum_{k=1}^{N}W_{k,\,\mathrm{self}}^\epsilon + \frac{\kappa}{2}\sum_{k=1}^{N}\sum_{j=1,\,j\neq k}^{N}\frac{q_j q_k}{|\mathbf{r}_j-\mathbf{r}_k|} + \frac{1}{2}\sum_{k=1}^{N}q_k\psi_R(\mathbf{r}_k) + O(\epsilon). \tag{20}$$

Equation (20), in its turn, is inserted in Eq. (12) for the force; as shown, the self-energies do not depend on the charge positions; hence, although diverging in the limit $\epsilon \longrightarrow 0$, *they do not contribute to the force.* The rest is pretty straightforward, except that one has to be careful when differentiating the last term on the right of Eq. (20) with $k = i$: as is seen from Eq. (8), in this case $\mathbf{r}_i$ stands for *two* (and not one!) arguments of ψ_R, namely, $\psi_R(\mathbf{r}_i) \equiv \psi_R(\mathbf{r}_i, \mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N)$, and *both of them* have to be differentiated. Bearing this in mind, the expression for the force finally becomes

$$\mathbf{F}^i = -\kappa q_i\sum_{k=1,\,k\neq i}^{N}\frac{\partial}{\partial\mathbf{r}_i}\frac{q_k}{|\mathbf{r}_i-\mathbf{r}_k|} - \frac{1}{2}\left[\sum_{k=1}^{N}q_k\frac{\partial}{\partial\mathbf{r}_i}\psi_R(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}_k} + q_i\frac{\partial}{\partial\mathbf{r}}\psi_R(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}_i}\right] = \kappa q_i\sum_{k=1,\,k\neq i}^{N}q_k\frac{\mathbf{r}_i-\mathbf{r}_k}{|\mathbf{r}_i-\mathbf{r}_k|^3} \tag{21}$$

$$-\frac{1}{2}\left[\sum_{k=1}^{N} q_k \frac{\partial}{\partial \mathbf{r}_i}\psi_R(\mathbf{r})|_{\mathbf{r}=\mathbf{r}_k} + q_i \frac{\partial}{\partial \mathbf{r}}\psi_R(\mathbf{r})|_{\mathbf{r}=\mathbf{r}_i}\right].$$

This is the general result for the electrostatics which can be transformed further. Indeed, the direct substitution of the expression for ψ_R from Eq. (6) into Eq. (21) provides the force in the form

$$\mathbf{F}^i = -\kappa q_i \sum_{k=1,\,k\neq i}^{N} \frac{\partial}{\partial \mathbf{r}_i}\frac{q_k}{|\mathbf{r}_i - \mathbf{r}_k|}$$

$$-\frac{\kappa q_i}{2}\left[\sum_{k=1}^{N} q_k \frac{\partial}{\partial \mathbf{r}_i} G_R(\mathbf{r}, \mathbf{r}_i)|_{\mathbf{r}=\mathbf{r}_k} + \sum_{j=1}^{N} q_j \frac{\partial}{\partial \mathbf{r}} G_R(\mathbf{r}, \mathbf{r}_j)|_{\mathbf{r}=\mathbf{r}_i}\right] \quad (22)$$

$$= -\kappa q_i\left[\sum_{k=1,\,k\neq i}^{N} \frac{\partial}{\partial \mathbf{r}_i}\frac{q_k}{|\mathbf{r}_i - \mathbf{r}_k|} + \sum_{k=1}^{N} q_k \frac{\partial}{\partial \mathbf{r}} G_R(\mathbf{r}, \mathbf{r}_k)|_{\mathbf{r}=\mathbf{r}_i}\right],$$

and here we have used the symmetry property of Eq. (5) to obtain the second equality. To make the result even more physically transparent, we rewrite Eq. (22), in its turn, in the following way:

$$\mathbf{F}^i = -\kappa q_i \nabla\left\{\sum_{k=1}^{N} q_k\left[\frac{1}{|\mathbf{r}-\mathbf{r}_k|} + G_R(\mathbf{r}, \mathbf{r}_k)\right] - \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}\right\}\Bigg|_{\mathbf{r}=\mathbf{r}_i} = -q_i \nabla\left[\psi(\mathbf{r}) - \frac{\kappa q_i}{|\mathbf{r}-\mathbf{r}_i|}\right]\Bigg|_{\mathbf{r}=\mathbf{r}_i}. \quad (23)$$

Note that the last expression, indeed, coincides with our intuitive conjecture about the form of the force.

IV. DISCUSSION

Our first remark on the expressions for the force in Eqs. (21)–(23) is that for the charges in a free space (volume D is the whole space, no boundaries are present), apparently, $G_R(\mathbf{r}, \mathbf{r}_k) \equiv 0$, $\psi_R \equiv 0$, and the classical Coulomb formula for the force is restored.

Next, Eq. (23) shows that the rule "the force is the charge times the field it is placed in" does work *if one counts the regular part of the field produced by the charge in question as a part of the "field the charge is placed in."* It also allows for a certain "minimal principle"; namely, to get the right answer for the force, one should remove from the field only the part which otherwise makes the result infinite, *and nothing beyond that.* As we mentioned in the Introduction, this result is supported by physical intuition. It becomes even more obvious if one notes that the singular part of the field removed is radial, and a radial field produces no force.

The contribution of the regular part of the field created by a charge to the force acting on it is especially important in the case of a single charge, as one may see from the simplest example of a charge near a conducting plane. It is precisely the regular part of the field produced by the charge in question (equal to the field of the image charge) that gives the whole answer when no other charges are present. Finally, an important question is how *robust* our regularization of the problem is, i.e., whether the result for the force changes or not if one uses a different regularization. There are two significant points demonstrating such robustness.

The first one is concerned with the *geometrical* regularization that we used. If one chooses domain D_k^ϵ around q_k to be not a ball but some differently shaped volume bounded by a smooth surface S_k^ϵ ("topological ball"), then it is not difficult to see that all the terms in Eq. (20) for the regularized energy remain unchanged, and, hence, our result for the force is still valid. This can be demonstrated in exactly the same way as above, only the computation of the integral over the surface S_k^ϵ in Eq. (19) requires a well-known result from potential theory (cf. [13], n. 193, or [14]).

As for the first integral on the right of Eq. (17), which defines the self-energy $W^\epsilon_{k,\,\mathrm{self}}$, its explicit expression is not required, and its only relevant property, namely, its independence of $\mathbf{r}_k$, is obvious.

An alternative way of regularization, so widely used during the whole "pre-Dirac delta-function" era, is *physical* regularization, where the point charge q_k is replaced, within a small volume D_k^ϵ, with some smooth charge distribution of the density $\rho_k^\epsilon(\mathbf{k})$ and the same total charge q_k, and ϵ is taken to be zero in the answer. From a technical point of view, this approach proves to be more complicated in this particular case, but it leads again to the same terms in Eq. (20) for the regularized energy. The key point here is to start with the following expression for the regularized energy,

$$W_D^\epsilon \equiv \frac{1}{2}\int_D \rho^\epsilon \psi\, dV = \frac{1}{2}\sum_{k=1}^{N}\int_{D_k^\epsilon} \rho_k^\epsilon \psi\, dV, \quad (24)$$

and then, instead of Eq. (3), split the potential into a sum of volume potentials of $\rho_k^\epsilon(\mathbf{r})$ over D_k^ϵ (which becomes singular in the limit) and a regular addition $\psi_R^\epsilon(\mathbf{r})$.

In particular, this regularization is used by Smythe in Section 3.08 of [6] for calculating the force on a single point charge in a domain with zero potential at the boundary. In that work, derivation is at the "physical level of accuracy" and the answer is not brought down to its physically most relevant form of Eq. (23). Moreover, the final answer [right-hand side of Eq. (2) in that section] is, unfortunately, formally diverging because of the inappropriate use of the notation for the total potential in the place where its regular part should be.

Finally, we want to end our discussion by mentioning that the electrostatic problem we just solved, as well as its generalizations (see Section 5), involve only volume charges. On the other hand, magnetostatic problems that deal, for example, with magnetic fluxes trapped in superconducting media (cf. [15]) give rise to surface charges. Analysis of these is of extreme importance for modern experimental physics [16]. No easy solution for the force between surface charges should be anticipated since the details of the boundary shape, such as its curvature, are expected to play a role; the interaction of such surface charges will be discussed elsewhere.

V. GENERALIZATION: OTHER BOUNDARY CONDITIONS

We can now generalize our result for other conditions at the boundary.

A modest but potentially useful generalization may be applied to the case of electrodes, where an arbitrary distribution of the potential $V(\mathbf{r})$, and not just a zero, is specified at the boundary:

$$\psi|_S = V(\mathbf{r}), \quad \mathbf{r} \in S. \tag{25}$$

Let us split the potential in two,

$$\psi = \psi^{(1)} + \psi^{(2)}, \tag{26}$$

in which the first is caused by point charges without any voltage applied to the boundary, and the second is entirely due to the boundary voltage. Therefore, $\psi^{(1)}$ satisfies the boundary value problem of Eqs. (1) and (2),

$$\Delta\psi^{(1)} = -\frac{1}{\varepsilon_0}\sum_{i=1}^{N} q_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \mathbf{r}, \mathbf{r}_i \in D; \tag{27}$$

$$\psi^{(1)}|_S = 0. \tag{28}$$

According to what was proven above, the force on a charge from the field specified by the potential $\psi^{(1)}$ is given according to Eq. (23),

$$\mathbf{F}^i_{(1)} = -q_i\nabla\left[\psi^{(1)}(\mathbf{r}) - \frac{\kappa q_i}{|\mathbf{r}-\mathbf{r}_i|}\right]\Bigg|_{\mathbf{r}=\mathbf{r}_i}. \tag{29}$$

On the other hand, the potential $\psi^{(2)}$, satisfying

$$\Delta\psi^{(2)} = 0, \quad \mathbf{r} \in D; \quad \psi^{(2)}|_S = V(\mathbf{r}), \tag{30}$$

describes a field *external* to all point charges, since it does not depend on them and their positions. Therefore, the force exerted by this field is

$$\mathbf{F}^i_{(2)} = -q_i\nabla\psi^{(2)}(\mathbf{r})|_{\mathbf{r}=\mathbf{r}_i}. \tag{31}$$

Using the superposition principle, we add these two forces to reinstate the result of Eq. (23) in the considered case:

$$\mathbf{F}^i = \mathbf{F}^i_{(1)} + \mathbf{F}^i_{(2)} = -q_i\nabla\left[\psi(\mathbf{r}) - \frac{\kappa q_i}{|\mathbf{r}-\mathbf{r}_i|}\right]\Bigg|_{\mathbf{r}=\mathbf{r}_i}. \tag{32}$$

The mixed boundary conditions

$$\psi|_{S_1} = V(\mathbf{r}), \quad \varepsilon_0\frac{\partial\psi}{\partial n}\Big|_{S_2} = \sigma(\mathbf{r}), \tag{33}$$

where the surfaces S_1, S_2 are nonintersecting ($S_1 \cap S_2 = \varnothing$) and comprise the whole boundary ($S_1 \cup S_2 = S$), and $V(\mathbf{r})$, $\sigma(\mathbf{r})$ are given functions, lead to the same standard result for the force [Eq. (23)] without any new technical difficulties. Indeed, we split the total potential in two as in Eq. (26) and require that

$$\Delta\psi^{(1)} = -\frac{1}{\varepsilon_0}\sum_{i=1}^{N} q_i\delta(\mathbf{r}-\mathbf{r}_i), \quad \mathbf{r}, \mathbf{r}_i \in D; \tag{34}$$

$$\psi^{(1)}|_{S_1} = 0, \quad \frac{\partial\psi^{(1)}}{\partial n}\Big|_{S_2} = 0 \tag{35}$$

and

$$\Delta\psi^{(2)} = 0, \quad \mathbf{r} \in D; \tag{36}$$

$$\psi^{(2)}|_{S_1} = V(\mathbf{r}), \quad \varepsilon_0\frac{\partial\psi^{(2)}}{\partial n}\Big|_{S_2} = \sigma(\mathbf{r}). \tag{37}$$

The derivation of the force from $\psi^{(1)}$ is performed exactly as in Section 3 and leads to Eq. (29). The field that is external to the charges from $\psi^{(2)}$ produces the force of Eq. (31), so by superposition the total force is again the same as in Eq. (23) [or Eq. (32)].

The appropriate splitting of the potential into two parts [Eq. (26)] is somewhat more difficult for the Neumann boundary condition,

$$\varepsilon_0\frac{\partial\psi}{\partial n}\Big|_S = \sigma(\mathbf{r}), \int_S \sigma(\mathbf{r})dA + Q = 0, \quad Q \equiv \sum_{j=1}^{N} q_j; \tag{38}$$

namely, the solvability criterion (the total charge must be zero) requires that, when splitting the potential, another charge Q (equal to the sum of the point charges q_i) be added and subtracted at some point $\mathbf{r}_*$ of the domain D to obtain

$$\Delta\psi^{(1)} = -\frac{1}{\varepsilon_0}\left[\sum_{i=1}^{N} q_i\delta(\mathbf{r}-\mathbf{r}_i) - Q\delta(\mathbf{r}-\mathbf{r}_*)\right], \quad \mathbf{r}, \mathbf{r}_i, \mathbf{r}_* \in D; \tag{39}$$

$$\frac{\partial\psi^{(1)}}{\partial n}\Big|_S = 0, \tag{40}$$

as well as

$$\Delta\psi^{(2)} = -\frac{Q}{\varepsilon_0}\delta(\mathbf{r}-\mathbf{r}_*), \quad \mathbf{r}, \mathbf{r}_* \in D; \tag{41}$$

$$\varepsilon_0 \frac{\partial\psi^{(2)}}{\partial n}\bigg|_S = \sigma(\mathbf{r}) \tag{42}$$

with both problems solvable. Again, the derivation of the force from $\psi^{(1)}$ satisfying the homogeneous boundary condition is performed exactly as before and leads to Eq. (29), the field $\psi^{(2)}$ external to the charges exerts the force given in Eq. (31), and the result of Eq. (23) holds by superposition. The problem itself, though, is not too realistic, except for the case of an insulated boundary, $\sigma(\boldsymbol{r}) \equiv 0$.

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