Whence the Force of $F = ma$? III: Cultural Diversity

Frank Wilczek

The concept of force, as we have seen, defines a culture. In the previous columns of this series (PHYSICS TODAY, October 2004, page 11, and December 2004, page 10) I've indicated how $F = ma$ acquires meaning through interpretation of—that is, additional assumptions about—F. This body of interpretation is a sort of folklore. It contains both approximations that we can derive, under appropriate conditions, from modern foundations, and also rough generalizations (such as "laws" of friction and of elastic behavior) abstracted from experience.

In the course of that discussion it became clear that there is also a smaller, but nontrivial, culture around m. Indeed, the conservation of m for ordinary matter provides an excellent, instructive example of an emergent law. It captures in a simple statement an important consequence of broad regularities whose basis in modern fundamentals is robust but complicated. In modern physics, the idea that mass is conserved is drastically false. A great triumph of modern quantum chromodynamics (QCD) is to build protons and neutrons, which contribute more than 99% of the mass of ordinary matter, from gluons that have exactly zero mass, and from u and d quarks that have very small masses. To explain from a modern perspective why conservation of mass is often a valid approximation, we need to invoke specific, deep properties of QCD and quantum electrodynamics (QED), including the dynamical emergence of large energy gaps in QCD and the smallness of the fine structure constant in QED.

Isaac Newton and Antoine Lavoisier knew nothing of all this, of course. They took conservation of mass as a fundamental principle. And they were right to do so, because by adopting that principle they were able to make brilliant progress in the analysis of motion and of chemical change. Despite its radical falsity, their principle was, and still is, an adequate basis for many quantitative applications. To discard it is unthinkable. It is an invaluable cultural artifact and a basic insight into the way the world works despite—indeed, in part, because of—its emergent character.

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The culture of *a*

What about a? There's a culture attached to acceleration, as well. To obtain a, we are instructed to consider the change of the position of a body in space as a function of time, and to take the second derivative. This prescription, from a modern perspective, has severe problems.

In quantum mechanics, bodies don't have definite positions. In quantum field theory, they pop in and out of existence. In quantum gravity, space is fluctuating and time is hard to define. So evidently serious assumptions and approximations are involved even in making sense of a's definition.

Nevertheless, we know very well where we're going to end up. We're going to have an emergent, approximate concept of what a body is. Physical space is going to be modeled mathematically as the Euclidean three-dimensional space $\mathbf{R}^3$ that supports Euclidean geometry. This tremendously successful model of space has been in continuous use for millennia, with applications in surveying and civil engineering that even predate Euclid's formalization.

Time is going to be modeled as the one-dimensional continuum $\mathbf{R}^1$ of real numbers. This model of time, at a topological level, goes into our primitive intuitions that divide the world into past and future. I believe that the metric structure of time—that is, the idea that time can be not only ordered but divided into intervals with definite numerical magnitude—is a much more recent innovation. That idea emerged clearly only with Galileo's use of pendulum clocks (and his pulse!).

The mathematical structures involved are so familiar and fully developed that they can be, and are, used routinely in computer programs. This is not to say they are trivial. They most definitely aren't. The classical Greeks agonized over the concept of a continuum. Zeno's famous paradoxes reflect these struggles. Indeed, Greek mathematics never won through to comfortable algebraic treatment of real numbers. Continuum quantities were always represented as geometric intervals, even though that representation involved rather awkward constructions to implement simple algebraic operations.

The founders of modern analysis (René Descartes, Newton, Gottfried Wilhelm Leibniz, Leonhard Euler, and others) were on the whole much more freewheeling, trusting their intuition while manipulating infinitesimals that lacked any rigorous definition. (In his *Principia*, Newton did operate geometrically, in the style of the Greeks. That is what makes the *Principia* so difficult for us to read today. The *Principia* also contains a sophisticated discussion of derivatives as limits. From that discussion I infer that Newton and possibly other early analysts had a pretty good idea about what it would take to make at least the simpler parts of their work rigorous, but they didn't want to slow down to do it.) Reasonable rigor, at the level commonly taught in mathematics courses today—the much-bemoaned epsilons and deltas—entered into the subject in the 19th century.

"Unreasonable" rigor entered in the early 20th century, when the fundamental notions from which real numbers and geometry are constructed were traced to the level of set theory and ultimately symbolic logic. In their *Principia Mathematica* Bertrand Russell and Alfred Whitehead develop 375 pages of dense mathematics before proving $1 + 1 = 2$. To be fair, their treatment could be slimmed down considerably if attaining that particular result were the ultimate goal. But in any case, an adequate definition of real numbers from symbolic logic involves some hard, complicated work. Having the integers in hand, you then have to define rational numbers and their ordering. Then you must complete them by filling in the holes so that any bounded increasing sequence has a limit. Then finally—this is the hardest part—you must demonstrate that the resulting system supports algebra and is consistent.

Perhaps all that complexity is a hint that the real-number model of space and time is an emergent concept that some day will be derived from physically motivated primitives that are logically simpler. Also, scrutiny of the construction of real numbers suggests natural variants, notably John Conway's surreal numbers, which include infinitesimals (smaller than any rational number!) as legitimate quantities.[1] Might such quantities, whose formal properties seem no less natural and elegant than those of ordinary real numbers, help us to describe nature? Time will tell.

Even the unreasonable rigor of symbolic logic does not reach ideal strictness. Kurt Gödel demonstrated that this ideal is unattainable: No reasonably complex, consistent axiomatic system can be used to demonstrate its own consistency.

But all the esoteric shortcomings in defining and justifying the culture of a clearly arise on an entirely different level from the comparatively mundane, immediate difficulties we have in doing justice to the culture of F. We can translate the culture of a, without serious loss, into C or FORTRAN. That completeness and precision give us an inspiring benchmark.

The computational imperative

Before they tried to do it, most computer scientists anticipated that to teach a computer to play chess like a grand master would be much more challenging than to teach one to do mundane tasks like drive a car safely. Notoriously, experience has proved otherwise. A big reason for that surprise is that chess is algorithmic, whereas driving a car is not. In chess the rules are completely explicit; we know very concretely and unambiguously what the degrees of freedom are and how they behave. Car driving is quite different: Essential concepts like "other driver's expectations" and "pedestrian," when you start to analyze them, quickly burgeon into cultures. I wouldn't trust a computer driver in Boston's streets because it wouldn't know how to interpret the mixture of intimidation and deference that human drivers convey by gestures, maneuvers, and eye contact.

The problem with teaching a computer classical mechanics is, of course, of more than academic interest: We'd like robots to get around and manipulate things; computer gamesters want realistic graphics; engineers and astronomers would welcome smart silicon collaborators—up to a point, I suppose.

The great logician and philosopher Rudolf Carnap made brave, pioneering attempts to make axiomatic systems for elementary mechanics, among many other things.[2] Patrick Hayes issued an influential paper, "Naive Physics Manifesto," challenging artificial-intelligence researchers to codify intuitions about materials and forces in an explicit way.[3] Physics-based computer graphics is a lively, rapidly advancing endeavor, as are several varieties of computer-assisted design. My MIT colleagues Gerald Sussman and Jack Wisdom have developed an intensely computational approach to mechanics,[4] supported every step of the way with explicit programs. The time may be ripe for a powerful synthesis, incorporating empirical properties of specific materials, successful known designs of useful mechanisms, and general laws of mechanical behavior into a fully realized computational culture of $F = ma$. Functioning robots might not need to know a lot of mechanics explicitly, any more than most human soccer players do; but *designing* a functioning robotic soccer player may be a job that can best be accomplished by a very smart and knowledgeable man-machine team.

Blur and focus

An overarching theme of this series has been that the law $F = ma$, which is sometimes presented as the epitome of an algorithm describing nature, is actually not an algorithm that can be applied mechanically (pun intended). It is more like a language in which we can easily express important facts about the world. That's not to imply it is without content. The content is supplied, first of all, by some powerful general statements in that language—such as the zeroth law, the momentum conservation laws, the gravitational force law, the necessary association of forces with nearby sources—and then by the way in which phenomenological observations, including many (though not all) of the laws of material science can be expressed in it easily.

Another theme has been that $F = ma$ is not in any sense an ultimate truth. We can understand, from modern foundational physics, how it arises as an approximation under wide but limited circumstances. Again, that does not prevent it from being extraordinarily useful; indeed, one of its primary virtues is to shield us from the unnecessary complexity of irrelevant accuracy!

Viewed this way, the law of physics $F = ma$ comes to appear a little softer than is commonly considered. It really does bear a family resemblance to other kinds of laws, like the laws of jurisprudence or of morality, wherein the meaning of the terms takes shape through their use. In those domains, claims of ultimate truth are wisely viewed with great suspicion; yet nonetheless we should actively aspire to the highest achievable level of coherence and explicitness. Our physics culture of force, properly understood, has this profoundly modest but practically ambitious character. And once it is no longer statuized, put on a pedestal, and seen in splendid isolation, it comes to appear as an inspiring model for intellectual endeavor more generally.

References

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