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Minimal subtraction and the Callan–Symanzik equation

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Abstract

The usual proof of renormalizability using the Callan–Symanzik equation makes explicit use of normalization conditions. It is shown that demanding that the renormalization group functions take the form required for minimal subtraction allows one to prove renormalizability using the Callan–Symanzik equation, without imposing normalization conditions. Scalar field theory and quantum electrodynamics are treated. © 1999 Elsevier Science B.V.

An elegant and compact proof of the perturbative renormalizability of scalar field theory can be given using the Callan–Symanzik equation [1,2]. This proof constructs the renormalization group functions and renormalized correlation functions order-by-order without ever encountering an infinite quantity. The steps in this proof are as follows:

(1) One proves that there is a skeleton expansion for Green functions—this is the step where the renormalizability of the theory appears.

(2) One selects normalization conditions in order to define the renormalization parts; in other words, the normalization conditions specify the finite renormalizations that must be fixed order by order in the perturbative subtraction of infinities. *The aim of this paper is to fix these finite renormalizations in an entirely different manner, as we explain below.* These normalization conditions are in one-to-one correspondence with independent superficially divergent irreducible proper vertices (i.e. divergent vertices that are not related to other divergences by Ward identities).

(3) Using the normalization conditions at every step to fix the finite renormalizations, one establishes simultaneously the existence of the Callan–Symanzik equation, and the existence of finite Green functions, order by order in the coupling constant,

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using differentiations with respect to the mass parameter to reduce the degree of divergence of any given irreducible vertex.

While this method is usually applied directly in four dimensions, it is easy to extend it to dimensions 'near' four, as described explicitly in [2].

The drawback in this proof is that it makes explicit use of normalization conditions. This makes it awkward to establish the existence of renormalized Ward identities in theories with non-linear symmetries, whereas these renormalized identities are trivially obtained, for theories in which dimensional continuation preserves the symmetries, if one uses minimal subtraction [2]. (There are, of course, theories in which dimensional regularization does not preserve such symmetries. In these cases the methods of the present paper do not apply.) Of course, on general grounds, one knows that there is a finite renormalization that takes one from the renormalized theory with normalization conditions to the renormalized theory with minimal subtraction of infinities. However, it should be possible to show the renormalizability with minimal subtraction directly, instead of via the construction of the finite renormalization. This is the aim of the present paper.

There appears to be no way to avoid using dimensional continuation to do what we wish to do—in essence, if we eschew the use of normalization conditions, some additional information is required to fix the finite renormalizations. In our proof, these finite renormalizations are fixed by requiring specific forms for the renormalization group functions, as functions of the dimensional continuation parameter. Notice that we never use dimensional *regularization*, so we are still dealing only with finite quantities, in accord with the approach of [1].

In order to avoid confusion, we wish to emphasize that there is an *entirely different* approach to using the renormalization group to prove renormalizability, which uses the Wilsonian renormalization group [3]. The relationship between the Callan–Symanzik equation approach and the Wilsonian approach is discussed in [4].

We first consider Euclidean ϕ^4 theory in $4-\epsilon$ dimensions. $\Gamma^{n,l}(p_i;q_j)$ denotes the renormalized 1PI n-point function with momenta p_i, and l insertions of $-\frac{1}{2}\phi^2(q_j)$. To be precise, we show that it is possible to compute, order by order, renormalized correlation functions which satisfy the Callan–Symanzik equation [2]

$$\begin{aligned}&\left[m\frac{\partial}{\partial m}+(\beta(g)-\epsilon g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)-l\eta_2(g)\right]\Gamma^{n,l}(p_1,\dots,p_n;q_1,\dots,q_l)\\&\quad=m^2(2+\delta(g))\Gamma^{n,l+1}(p_1,\dots,p_n;q_1,\dots,q_l,0),\end{aligned}\tag{1}$$

with β,η,η_2,δ power series in the coupling g alone, and therefore finite order by order. This form of the renormalization group functions ensures that the renormalization constants are Laurent series in $1/\epsilon$, with no finite pieces, as is appropriate for minimal subtraction [2]. In this scheme, the normalization conditions are replaced by

$$\Gamma^{2,0}(p=0)=m^2(1+a),$$

$$\frac{\partial}{\partial p^2}\Gamma^{2,0}(p=0)=1+b,$$

$$\Gamma^{4,0}(p_i = 0) = gm^{\epsilon}(1 + c),$$
$$\Gamma^{2,1}(p_i = 0; q = 0) = 1 + d, \tag{2}$$

where $a(g,\epsilon), b(g,\epsilon), c(g,\epsilon)$, and $d(g,\epsilon)$, are power series, at least $O(g)$, which we shall show to be finite as $\epsilon \downarrow 0$. These three 1PI functions are the primitively divergent vertex functions in this model. We let A_r stand for any quantity A computed up to order g^r in the perturbative expansion.

We write Eq. (1) in the form

$$\left[m\frac{\partial}{\partial m} - \epsilon g\frac{\partial}{\partial g}\right]\Gamma^{n,l}_{r+1} = \left(\left[\frac{n}{2}\eta + l\eta_2 - \beta\frac{\partial}{\partial g}\right]\Gamma^{n,l}\right)_{r+1} + m^2\left((2+\delta)\Gamma^{n,l+1}\right)_{r+1}. \tag{3}$$

The proof proceeds by induction, so it is important to make explicit the g dependence of all quantities at the lowest order. $\beta(g) = O(g^2)$, $\eta(g) = O(g^2)$, $\eta_2(g) = O(g)$, and $\delta(g) = O(g)$ [2]. Furthermore, $b = O(g^2)$, $a, c, d = O(g)$, $\Gamma^{4,1} = O(g^2)$, and $\Gamma^{2,2} = O(g)$, as can readily be seen from the lowest order diagrams.

The induction hypothesis is that the primitively divergent vertex functions have been rendered finite up to and including $O(g^r)$, except for $\Gamma^{4,0}$ which is assumed finite up to order $O(g^{r+1})$. This implies that a_r, b_r, c_r, and d_r are finite. Further, we assume that β_{r+1} is finite, as are $\eta_r, \eta_{2,r}, \delta_r$.

Consider Eq. (3) for $n = 4, l = 0$. Given the induction hypothesis and the fact that $\Gamma^{4,1}$ has a skeleton expansion, $\Gamma^{4,1}_{r+2}$ is finite. Then all terms on the r.h.s. of Eq. (3) are finite to $O(g^{r+2})$ if we can show that the combination $(2\eta g - \beta)_{r+2}$ is finite. To show this, we evaluate (3) at $p_i = 0 = q$, giving

$$(2\eta g - \beta)_{r+2} + \left(\epsilon g^2\frac{\partial}{\partial g}c\right)_{r+2} = C(\epsilon, g)_{r+2}, \tag{4}$$

where C_{r+2} is finite, and hence can be uniquely written as $C_{r+2} = (A(g) + \epsilon B(\epsilon, g))_{r+2}$, where A_{r+2} and B_{r+2} are finite. Thus, there exists a unique solution of (4) with c_{r+1} finite, and $(2\eta g - \beta)_{r+2}$ finite and ϵ independent (i.e., $(2\eta g - \beta)_{r+2} = A(g)_{r+2}$, $\left(g^2\partial_g c\right)_{r+2} = B(\epsilon, g)_{r+2}$). Of course, we know nothing of the finiteness of η_{r+1} or of β_{r+2} separately, but we do not need this information to integrate Eq. (3) for $n = 4, l = 0$ to obtain $\Gamma^{4,0}$ at arbitrary momenta. Indeed, we now see that

$$\left[m\frac{\partial}{\partial m} - \epsilon g\frac{\partial}{\partial g}\right]\Gamma^{4,0}_{r+2}(p_i; g, \epsilon) = gm^{\epsilon} f\left(\frac{p_i}{m}, g, \epsilon\right), \tag{5}$$

for some finite dimensionless function $f = O(g)$, with $\Gamma^{4,0}_{r+2}(0; g, \epsilon) = gm^{\epsilon}(1 + c_{r+1})$ finite. Since

$$\left[m\frac{\partial}{\partial m} - \epsilon g\frac{\partial}{\partial g}\right]\int_0^1 \frac{d\alpha}{\alpha} f\left(\alpha\frac{p_i}{m}, g\alpha^{\epsilon}, \epsilon\right) = -f\left(\frac{p_i}{m}, g, \epsilon\right)$$

for any function f regular at zero momentum and coupling, such that $f(0,0,\epsilon) = 0$, we have

$$\Gamma^{4,0}_{r+2}(p_i;g,\epsilon) = gm^{\epsilon}\left[1 - \int_0^1 \frac{\mathrm{d}\alpha}{\alpha} f\left(\alpha\frac{p_i}{m}, g\alpha^{\epsilon}, \epsilon\right)\right]. \tag{6}$$

As is standard [1,2], this assumes that the limit $p \downarrow 0$ does not introduce any pathologies into the integral over α, so that the finiteness already proved for c_{r+1} suffices to render Eq. (6) well defined. In perturbation theory, for $m > 0$, this regularity at low momenta is physically reasonable.

The next step in the proof requires showing the finiteness of $\Gamma^{2,1}$ to $O(g^{r+1})$. Using the skeleton expansion of $\Gamma^{2,2}$ and the induction hypothesis, Eq. (3) for $n = 2, l = 1$ has a r.h.s. which is finite to $O(g^{r+1})$ if the combination $(\eta + \eta_2)_{r+1}$ is finite. As above, we can evaluate at $p_i = 0 = q$ to give

$$-\epsilon g\frac{\partial}{\partial g}d = \left(\eta + \eta_2 - \beta\frac{\partial}{\partial g}d\right) + (\eta + \eta_2)d + (2 + \delta)m^2\Gamma^{2,2}(0;0). \tag{7}$$

This can be written as

$$(\eta + \eta_2)_{r+1} + \left(\epsilon g\frac{\partial}{\partial g}d\right)_{r+1} = D(\epsilon, g)_{r+1},$$

where $D(\epsilon, g)_{r+1}$ is finite, and thus we can conclude that we may uniquely take d_{r+1} to be finite, and $(\eta + \eta_2)_{r+1}$ to be finite and ϵ-independent. We can integrate Eq. (3) for $n = 2, l = 1$ since it now takes the form

$$\left[m\frac{\partial}{\partial m} - \epsilon g\frac{\partial}{\partial g}\right]\Gamma^{2,1}_{r+1}(p_i;q;g,\epsilon) = f_1\left(\frac{p_i}{m}, \frac{q}{m}, g, \epsilon\right),$$

with f_1 finite to $O(g^{r+1})$, given the finiteness of $(\eta + \eta_2)_{r+1}$ and d_{r+1}.

Having shown that $\Gamma^{2,1}$ is finite to the next order, we can now consider $\Gamma^{2,0}$. Eq. (3) for $n = 2, l = 0$ has a r.h.s. which is finite if η_{r+1} and δ_{r+1} are finite. First note that

$$\frac{\partial}{\partial p^2}\Gamma^{2,1}|_{p^2=0} = O(g^2).$$

Then Eq. (3) for $n = 2, l = 0$, gives, after differentiating with respect to p^2 at zero momentum,

$$\left(\eta + \epsilon g\frac{\partial}{\partial g}b\right)_{r+1} = E(\epsilon, g)_{r+1}, \tag{8}$$

where $E(\epsilon, g)_{r+1}$ is finite. As above, we deduce that there exists a unique solution of (8) with b_{r+1} finite and η_{r+1} finite and independent of ϵ. Then with the known finiteness of $(2\eta g - \beta)_{r+2}$, and $(\eta + \eta_2)_{r+1}$, we find that β is finite to $O(g^{r+2})$, with η_2 finite to $O(g^{r+1})$.

We now consider Eq. (3) for $n = 2, l = 0$ at $p = 0$, and find after subtracting Eq. (7),

$$2(a-d) - \epsilon g \frac{\partial}{\partial g}(a-d) = \eta(a-d) + (\delta - \eta_2)(1+d)$$
$$-\beta \frac{\partial}{\partial g}(a-d) - (2+\delta)\Gamma^{2,2}(0;0)m^2. \qquad (9)$$

Observe that there is no way to determine $\delta - \eta_2$ independent of $a - d$. This should be expected. In minimal subtraction $\delta = \eta_2$ is equivalent to $Z_m = Z_2$, (Z_m and Z_2 are the multiplicative renormalization constants for the mass and the $-\frac{1}{2}\phi^2$ insertion, respectively). In fact, in any scheme Z_m/Z_2 is a finite quantity [2]. In our present approach, we only deal with finite quantities, so we can consistently set $\delta = \eta_2$, thereby determining a unambiguously (since d is already known).[2] It is possible to let $\delta = \delta(g,\epsilon)$ and impose $a = b$, i.e. let m be the actual physical mass.

To complete our induction, we must exhibit a finite integral expression for $\Gamma^{2,0}$ to $O(g^{r+1})$. Having proven the finiteness of $\eta_{r+1}, \delta_{r+1}, a_{r+1}, b_{r+1}$, Eq. (3) for $n = 2, l = 0$ implies that $(\Gamma^{2,0}(p) - m^2(1+a) - p^2(1+b))_{r+1}$ satisfies an equation of the form

$$\left[m\frac{\partial}{\partial m} - \epsilon g \frac{\partial}{\partial g}\right]\left(\Gamma^{2,0}(p) - m^2(1+a) - p^2(1+b)\right)_{r+1} = m^2 \hat{f}\left(\frac{p}{m}; g, \epsilon\right), \qquad (10)$$

for a finite function $\hat{f} = O(p^4)$ for $|p|$ small. The integrated form of Eq. (10) requires showing that

$$\int_0^1 \frac{d\alpha}{\alpha^3} \hat{f}\left(\alpha \frac{p}{m}; g\alpha^{\epsilon}, \epsilon\right)$$

is finite, but this is obvious from the behaviour of $\hat{f}$ for $|p|$ small.

We have therefore completed the induction step, showing that the Callan–Symanzik equation can be used to prove the renormalizability of ϕ^4 theory in the minimal subtraction scheme, without ever imposing normalization conditions.

We turn now to an extension of this reasoning to the case of quantum electrodynamics. This was considered by Blaer and Young [5], but the existence of renormalized Ward identities seems to have been assumed without discussion in their work. In our formulation, since we use minimal subtraction, the existence of renormalized Ward identities is automatic. In the following, we show how the Ward identities constrain the renormalization group functions. The remaining steps then follow more or less as in Ref. [5], and are not reproduced here.

We follow the notation of Ref. [2], and consider the case of massive Euclidean QED, with m the mass of the photon, M the mass of the electron, and ξ the gauge parameter. The complete 1PI effective action, Γ, may be written as a sum $\Gamma[A, \psi, \bar{\psi}; e, m, M, \xi] = \Gamma_s + \frac{1}{2}\int(m^2A^2 + \xi^{-1}(\partial \cdot A)^2)$, where Γ_s satisfies the homogeneous equation

[2] There exists another solution of (9) for $(a-d)$ with the $O(g^{r+1})$ term having an essential singularity at $\epsilon = 0$. In our present framework this solution cannot be ruled out, but it prevents the induction from proceeding beyond tree level. Using the explicit form of dimensional regularization, this solution can be rejected because an essential singularity in ϵ cannot occur at any finite order in perturbation theory.

$$\left[\partial_\mu \frac{\delta}{\delta A_\mu} + ieM^{\epsilon/2}\left(\psi\frac{\delta}{\delta\psi} - \bar{\psi}\frac{\delta}{\delta\bar{\psi}}\right)\right]\Gamma_s = 0. \tag{11}$$

Thus Γ_s is gauge invariant. The general form of the Callan–Symanzik equation, differentiating with respect to M, is

$$\left[M\frac{\partial}{\partial M} + (\beta - \frac{\epsilon}{2}e)\frac{\partial}{\partial e} - \frac{n}{2}\eta_A - \frac{k}{2}\eta_\psi - l\eta_2 + \frac{\eta_m}{2}m\frac{\partial}{\partial m} + \alpha\xi\frac{\partial}{\partial\xi}\right]$$
$$\times\Gamma^{n,k,l}(p_1,\dots,p_n;r_1,\dots,r_k;q_1,\dots,q_l) = M(1+\delta)\Gamma^{n,k,l+1}(p_1,\dots,q_l,0), \tag{12}$$

relating the proper vertex with n photons, k electrons, and l insertions of $\bar{\psi}\psi$ to the proper vertex with one additional insertion of $\bar{\psi}\psi$ at zero momentum. There are seven independent renormalization group functions in this equation, $\alpha, \beta, \eta_A, \eta_\psi, \eta_m, \delta, \eta_2$, all functions of $e, \xi, m/M$, with no ϵ dependence.

For integrating Eq. (12) in a manner consistent with Eq. (11), it is necessary that the Callan–Symanzik equation be satisfied by Γ_s, not just by Γ. This implies $\eta_m = \eta_A$ and $\eta_A = -\alpha$. Further, commuting Eq. (12) with Eq. (11), we find $\eta_A = 2\beta/e$. Thus, we are left with four independent functions, and Eq. (12) simplifies to

$$\left[M\frac{\partial}{\partial M} - \epsilon e^2\frac{\partial}{\partial e^2} + \eta_A\left\{e^2\frac{\partial}{\partial e^2} + m^2\frac{\partial}{\partial m^2} - \xi\frac{\partial}{\partial\xi} - \frac{n}{2}\right\} - \frac{k}{2}\eta_\psi - l\eta_2\right]$$
$$\times\Gamma^{n,k,l}(p_1,\dots,q_l)$$
$$= M(1+\delta)\Gamma^{n,k,l+1}(p_1,\dots,q_l,0),$$

The form of the QED vertex functions at zero momentum are much restricted by Eq. (11),

$$\begin{aligned}
\Gamma^{2,0,0\mu\nu}(p=0) &= m^2\delta^{\mu\nu},\\
\frac{\partial}{\partial p^2}\Gamma^{2,0,0\mu}{}_{\mu}(p=0) &= (3-\epsilon)(1+a) - \frac{1}{\xi},\\
\Gamma^{0,2,0}(r=0) &= M(1+c),\\
\frac{\partial}{\partial r^\mu}\Gamma^{0,2,0}(r=0) &= i\gamma_\mu(1+b),\\
\Gamma^{0,2,1}(r_i=0;q=0) &= 1+d,\\
\Gamma^{1,2,0\mu}(p=0;r_i=0) &= ieM^{\epsilon/2}\gamma^\mu(1+b),
\end{aligned} \tag{13}$$

where a, b, c, d are all functions of $e, m/M, \xi$, *and* ϵ, finite as $\epsilon \downarrow 0$. Note that the presence of a photon mass ensures that there are no pathologies associated with these conditions, and that we may assume analyticity at zero-momentum. The rest of the analysis follows Ref. [5], with changes appropriate to minimal subtraction as discussed above in detail for scalar field theory.

In future work, we plan to apply this method of construction of renormalized gauge theories to non-abelian gauge theories, using the Curci–Ferrari action [6].

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