Bayesian statistics, Occam razor, and model-independent learning of continuous probability densities

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Joint work with:

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consistency	✓	✓
convergence rates	optimal	optimal
model selection	?	✓ (disagreement remains)
use of prior knowledge	✓	?

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$\dim \boldsymbol{\alpha} = K_A$
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(See: Bayes factors, Occam factors; Jaynes 1968, 1979)

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Does this generalize to infinite–dimensional models?

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Both settings hypersensitive to fluctuations in $F(t)$.
Smoothing is *required*.

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(See: Bialek, Callan, Strong, 1996)

Quantum Field Theory analogy

Fix ℓ and η:

$$= \frac{\langle Q(x)Q(x_1)\cdots Q(x_N)\rangle^0}{\underbrace{\langle Q(x_1)\cdots Q(x_N)\rangle^0}}$$

Correlation function in a QFT

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\langle Q \rangle &= \frac{\int [dQ]\, \mathcal{P}[Q]\, Q(x) \prod_{i=1}^{N} Q(x_i)}{\int [dQ]\, P[Q]\, \prod_{i=1}^{N} Q(x_i)} \\
&= \frac{\langle Q(x) Q(x_1) \cdots Q(x_N) \rangle^0}{\underbrace{\langle Q(x_1) \cdots Q(x_N) \rangle^0}}
\end{aligned}
$$

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Explicit form of correlation functions

$$\text{C. F.} \equiv \int [dQ]\mathcal{P}[Q]\prod_{i=1}^{N} Q(x_i)$$

$$= \int [d\phi]\,\frac{1}{\ell_0^N}\,\mathrm{e}^{-S[\phi]}\,\delta\left[\int dx \frac{1}{\ell_0}\mathrm{e}^{-\phi} - 1\right]$$

$$\underbrace{S[\phi]}_{\text{action}} = \underbrace{\frac{\ell^{2\eta-1}}{2}\int dx(\partial_x^\eta\phi)^2}_{\text{kinetic term}} + \underbrace{\sum_i \phi(x_i)}_{\text{random potential}}$$

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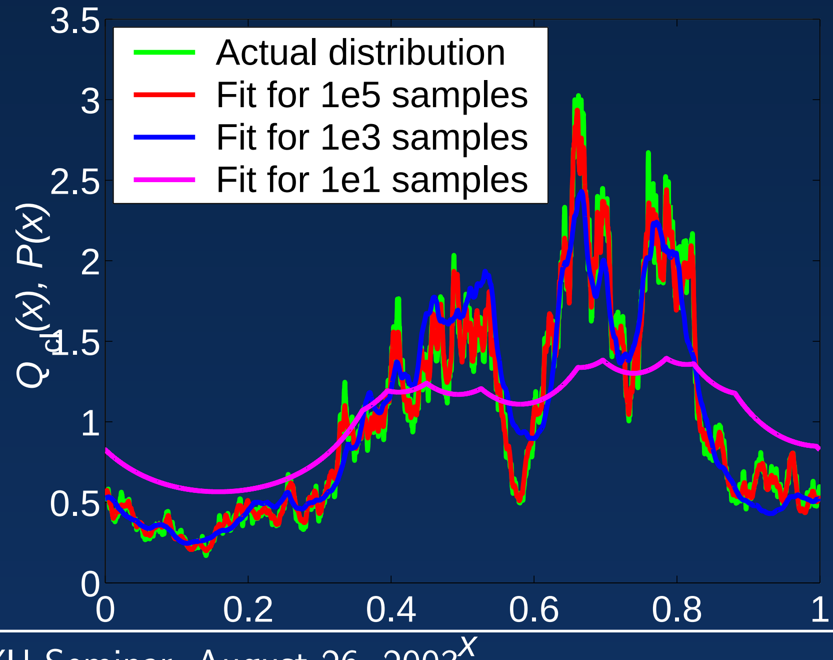

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&+ \underbrace{\frac{1}{2}\sqrt{\frac{N}{\ell\ell_0}}\int dx \mathrm{e}^{-\phi_{\mathrm{cl}}(x)/2}}_{\text{fluctuations, complexity, } error}
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For a different η:

$$\Lambda(N) \sim \left(\frac{L}{\ell}\right)^{1/2\eta} N^{1/2\eta-1}$$

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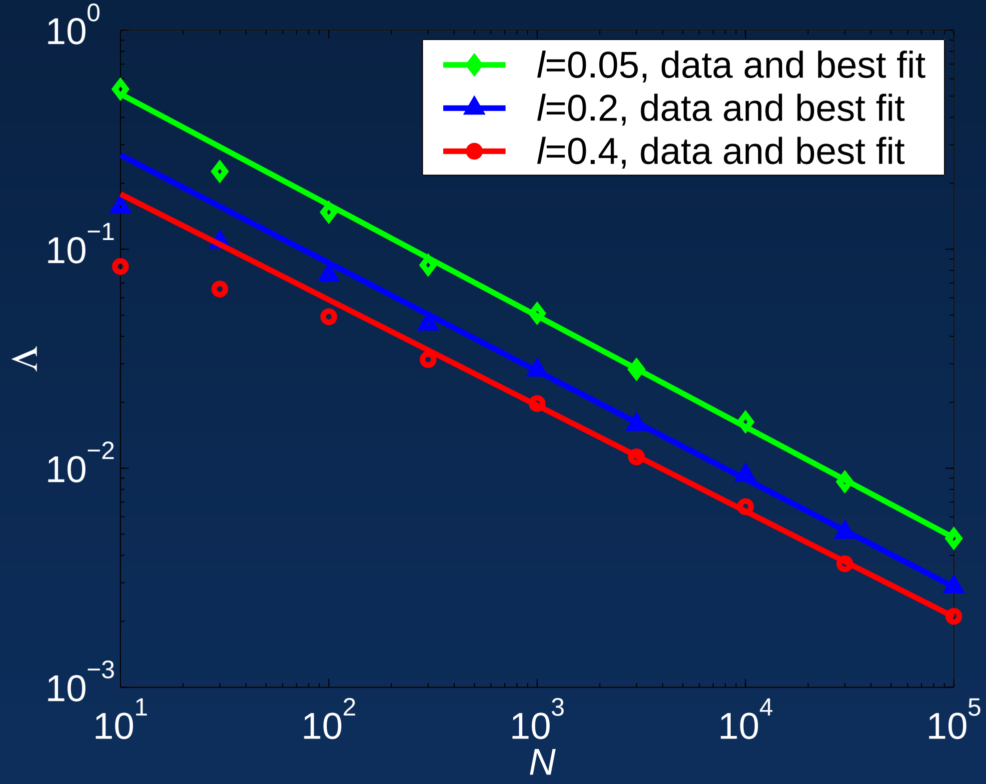
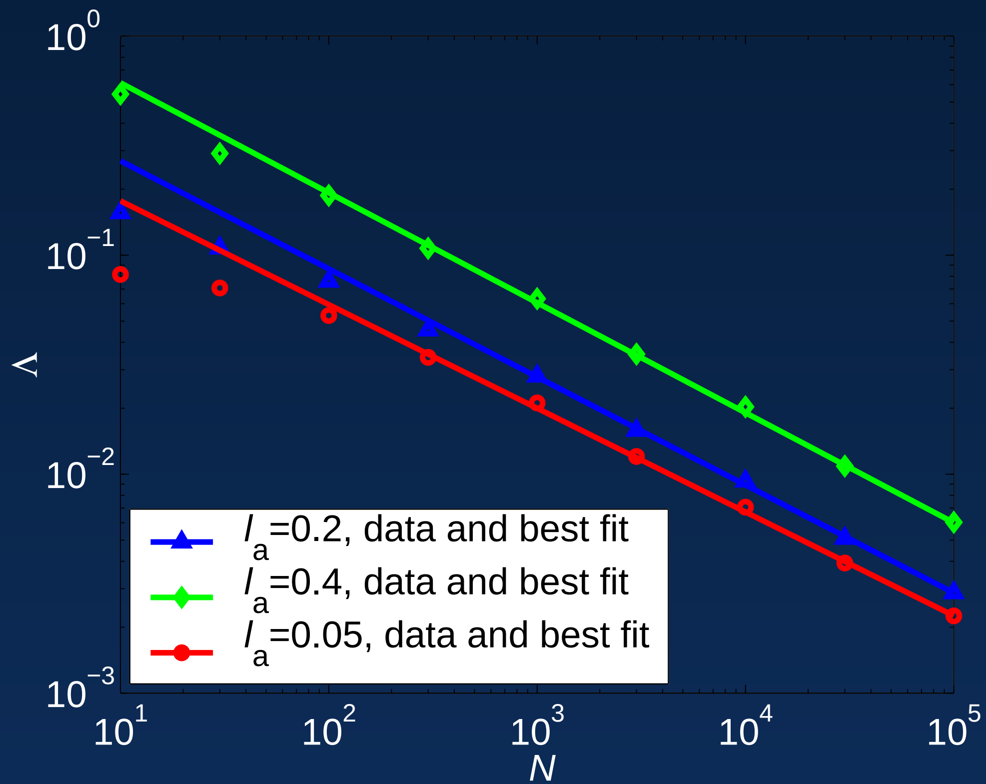
Note: we must have $\eta > 1/2$ for convergence of the integrals.

Learning typical cases



$$
\begin{aligned}
\ell = 0.4, &\quad \Lambda = (0.54 \pm 0.07) N^{-0.483 \pm 0.014} \\
\ell = 0.2, &\quad \Lambda = (0.83 \pm 0.08) N^{-0.493 \pm 0.09} \\
\ell = 0.05, &\quad \Lambda = (1.64 \pm 0.16) N^{-0.507 \pm 0.09}
\end{aligned}
$$

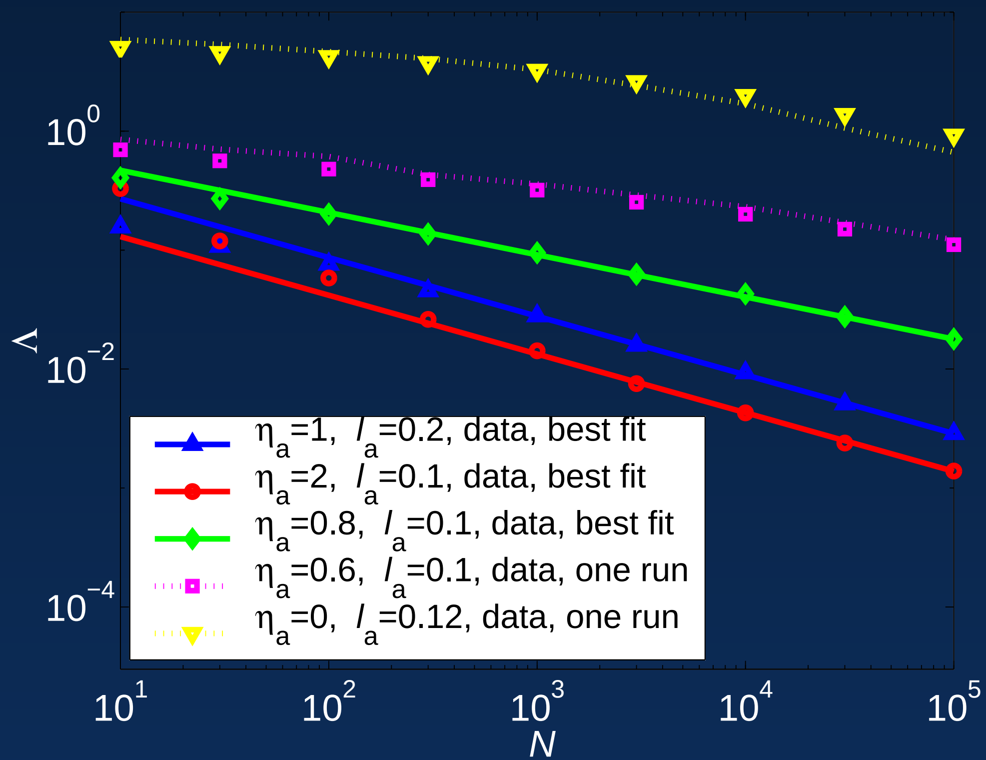
Learning marginal outliers



$$\ell_a = 0.4, \quad \Lambda = (0.56 \pm 0.08) N^{-0.477 \pm 0.015}$$
$$\ell_a = 0.05, \quad \Lambda = (1.90 \pm 0.16) N^{-0.502 \pm 0.008}$$

Learning at $\ell = 0.2$.

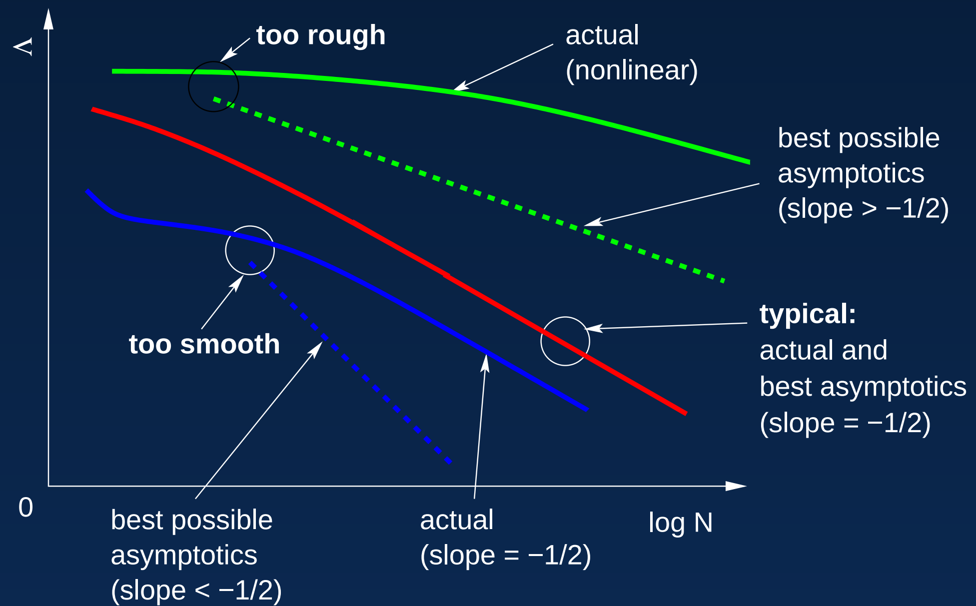
Learning strong outliers



$$\eta_a = 2,\ \ell_a = 0.1, \qquad \Lambda = (0.40 \pm 0.05) N^{-0.493 \pm 0.013}$$
$$\eta_a = 0.8,\ \ell_a = 0.1, \quad \Lambda = (1.06 \pm 0.08) N^{-0.355 \pm 0.008}$$

$\ell = 0.1$ for $\eta_a = 0$ and $\ell = 0.2$ otherwise

Conclusions for fixed η and ℓ



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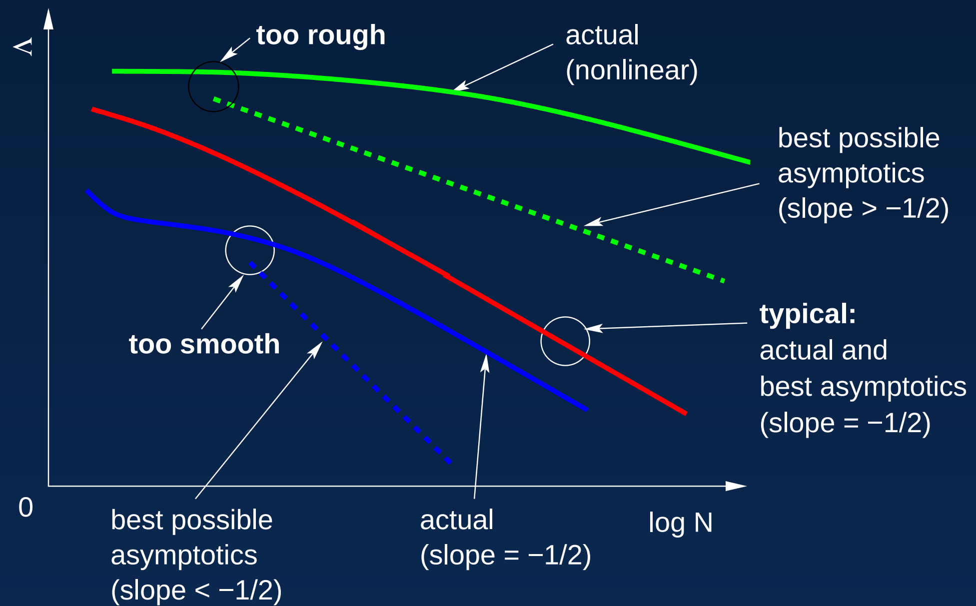

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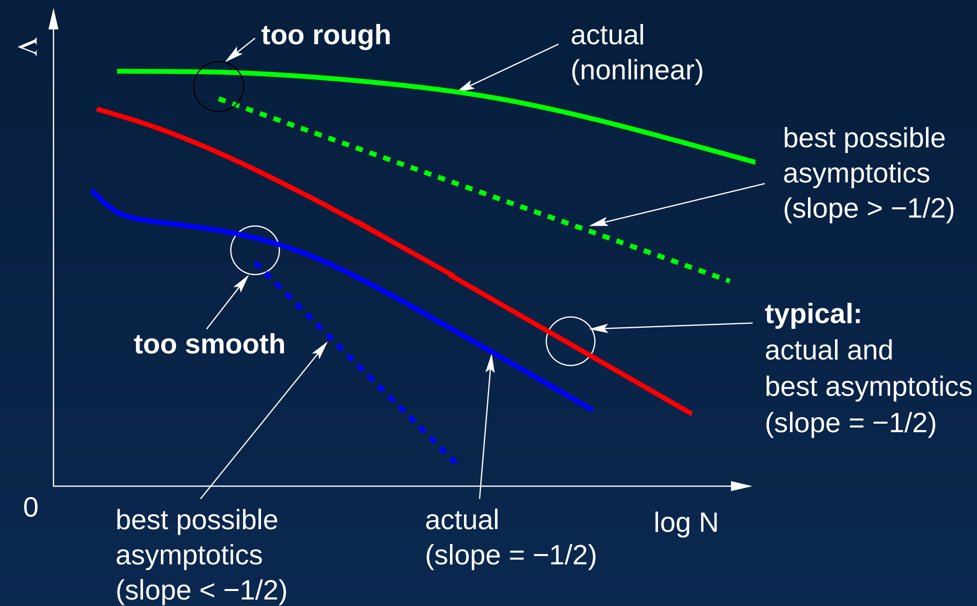

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- but suboptimal performance for learning outliers

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Some ℓ^* *always* dominates the C. F. and $\langle Q \rangle$!

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data $>$ smoothing	smoothing $>$ data

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Calculations: What is ℓ^* for η_a and ℓ_a?

If $\eta = \eta_a$, then $\ell^* = \ell_a$. Otherwise:

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best possible performance	better, but not best performance

Averaging over ℓ and allowing $\ell^* = \ell^*(N)$ deals with

qualitatively wrong smoothness $\eta_a \neq 1$!

Numerics: What is ℓ^* for η_a and ℓ_a?

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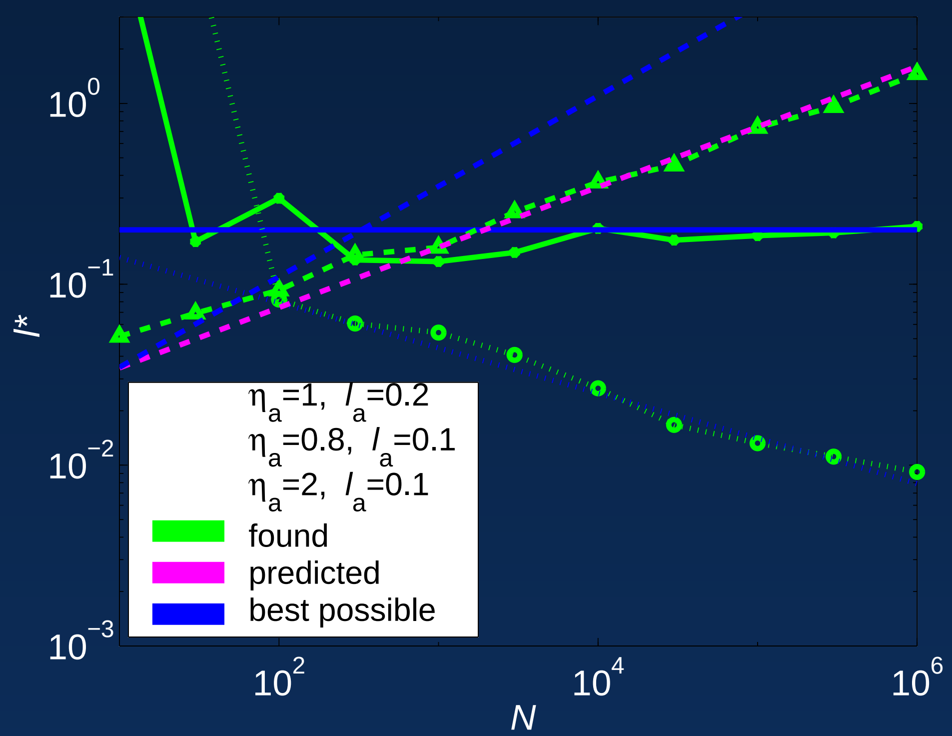
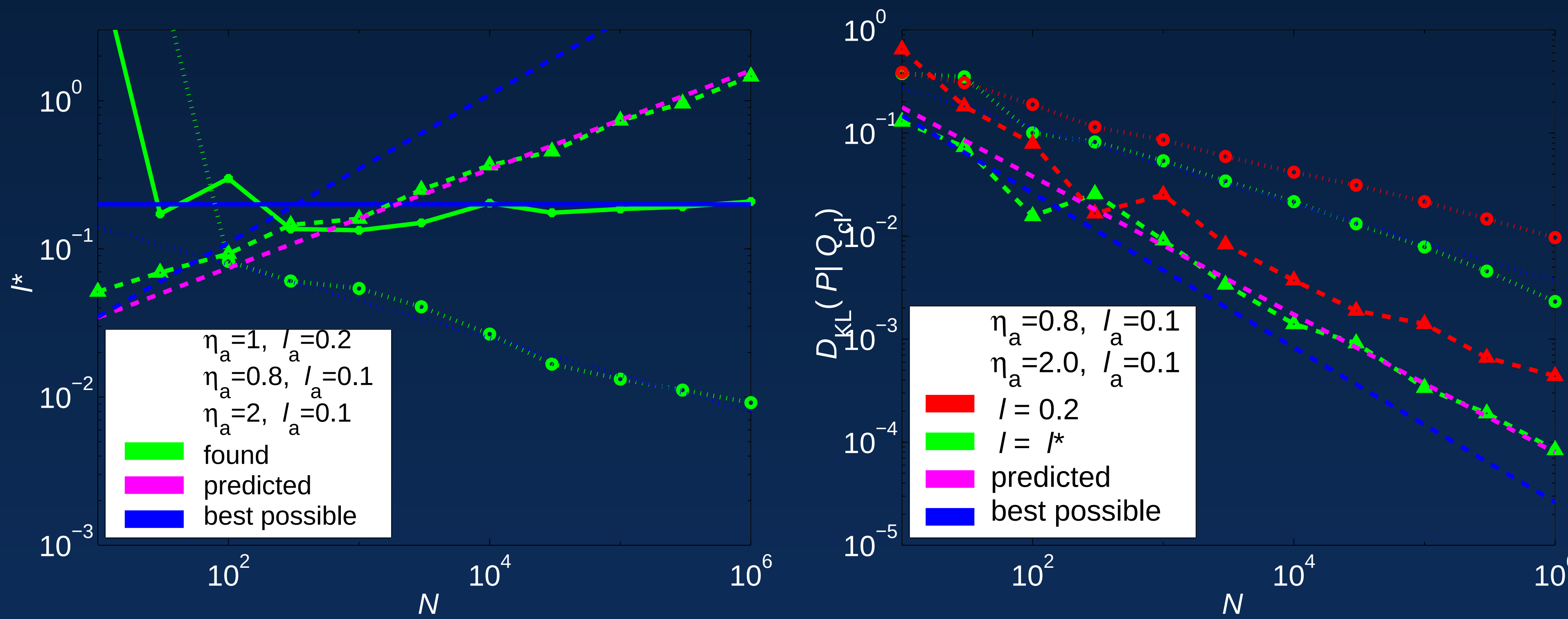


Note: just single runs shown.

Numerics: What is ℓ^* for η_a and ℓ_a?



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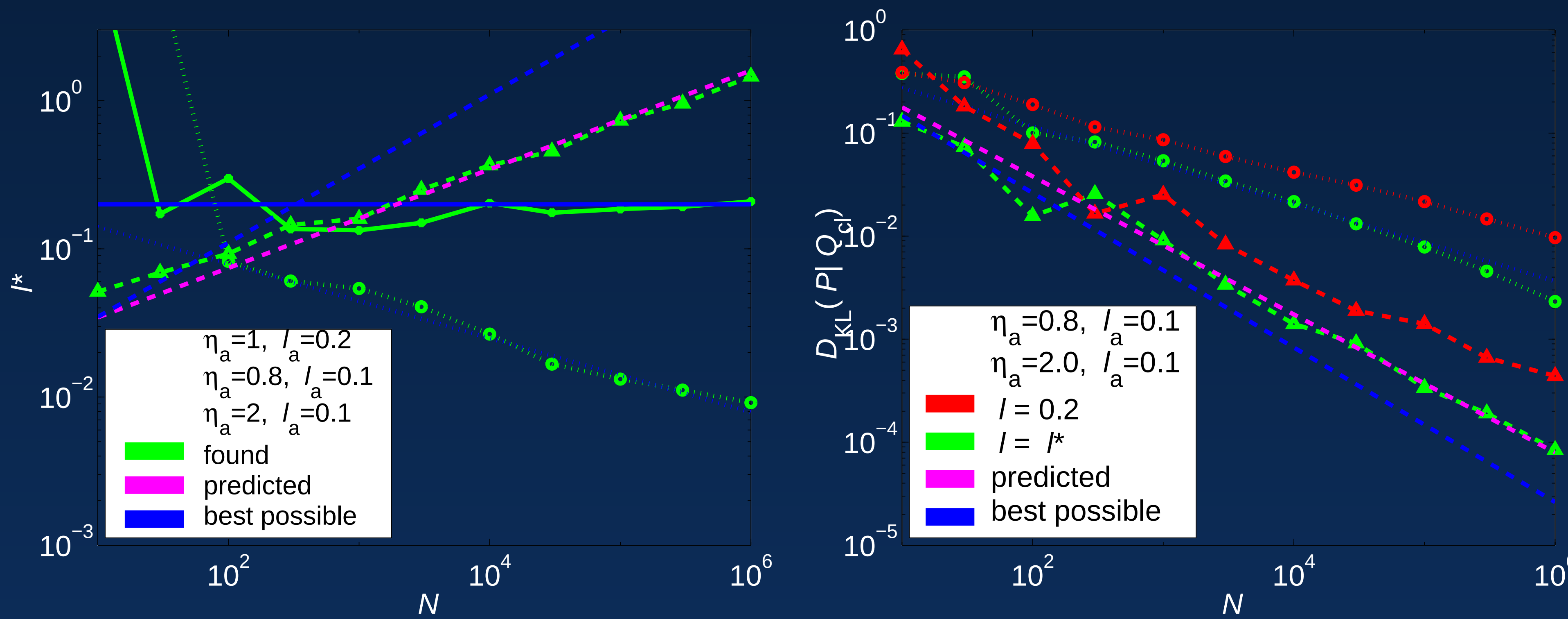

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Approaching model–independend optimal inference!

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- a lot in common with the Gaussian Processes theory; however normalization constraint is important

Summary

Bayesian smoothness (model) selection works for nonparametric spline priors!

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There is hope that all of this problems are resolvable in a single formulation.