Stochastic Path Integral Formulation of Full Counting Statistics

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We derive a stochastic path integral representation of counting statistics in semiclassical systems. The formalism is introduced on the simple case of a single chaotic cavity with two quantum point contacts, and then further generalized to find the propagator for charge distributions with an arbitrary number of counting fields and generalized charges. The counting statistics is given by the saddle-point approximation to the path integral, and fluctuations around the saddle point are suppressed in the semiclassical approximation. We use this approach to derive the current cumulants of a chaotic cavity in the hot-electron regime.

Noise properties of electrical conductors are interesting because they reveal additional information beyond linear response [1]. In the pioneering work of Levitov and Lesovik [2], the optics concept of full counting statistics (FCS) for photons was introduced for electrons in the context of mesoscopic physics. FCS gives the probability of counting a certain number of particles at a measurement apparatus in a certain amount of time and finds not only conductance and shot noise, but all higher current cumulants as well. Several methods have been used in finding this quantity. Originally, quantum mechanical methods based on scattering theory [2,3], the Keldysh approach put forth by Nazarov [4], or sigma models [5] have been advanced and have been successfully applied to a number of problems among which we mention only multiterminal structures [6], normal-superconducting samples [7], combined photon/electron statistics [8], and conductors which are current (instead of voltage) biased [9].

A quantum mechanical treatment of transport shows that the leading contribution to current cumulants is of the order of the channel number N. For many conductors or circuits of interest, this leading order is a classical quantity [10]. Weak localization or universal conductance fluctuations provide only a small correction of order 1. Clearly, it is desirable to have a purely classical theory to calculate classical results. To provide such a derivation of FCS is the main purpose of this work.

That a purely classical theory should be developed was realized by de Jong [11] who put forth a discussion for problems which can be described with the help of master equations [12]. A more general approach, leading to a set of rules for a cascade of higher order cumulants, was recently invented by Nagaev [13] and applied to chaotic cavities [14]. The work presented here aims at providing a foundation for the cascade approach by deriving a functional integral from which FCS, but also dynamical quantities such as correlation functions, can be obtained.

The approach provided here applies to an arbitrary mesoscopic network. Its classical nature does not allow

the treatment of weak localization corrections nor is it applicable to macroscopic quantum effects like the proximity effect. Our approach is based on a separation of time scales: fast microscopic fluctuations cause variations of conserved quantities (like the charge inside the conductor) on much longer time scales. Such a separation is present in normal-diffusive wires [13,15], superconducting-normal structures outside the proximity regime [16], as well as in stochastic problems beyond mesoscopic physics [17].

Path integral derivation.—To introduce our formalism, we consider transport through a chaotic cavity (see Fig. 1) which is a large conductor connected to metallic leads L and R through ideal point contacts. The leads are at different chemical potentials μ_R and $\mu_L = \mu_R + eV$, generating a current I. The number of modes in each point contact $N_{L,R}$ and the number of states eVn_F (n_F being a Fermi density of states in the cavity) are large, so that the transport is classical. We assume elastic transport (no energy relaxation), zero temperature, and chaotic electron dynamics in the cavity [18]. Then, the electron distribution in the cavity f_C is independent of the electron's
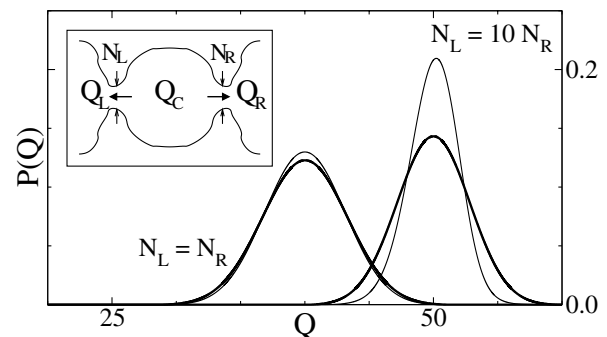

FIG. 1. Full counting statistics of a chaotic cavity at zero temperature: Comparison between hot (thick lines) and cold (thin lines) electron regime for both symmetric ($\langle Q \rangle = 40e$) and asymmetric ($\langle Q \rangle = 50e$) cases. Inset: The chaotic cavity with two point contacts.

coordinate, momentum, and energy in the interval $\mu_R < \varepsilon < \mu_L$ [19,20]. The average values of outgoing currents $I_{L,R}$ are given by $\langle I_R \rangle = (e^2 V/2\pi\hbar)N_R\langle f_C \rangle$ and $\langle I_L \rangle = -(e^2 V/2\pi\hbar)N_L(1 - \langle f_C \rangle)$. Since current is conserved, the average current through the cavity is $\langle I \rangle = (e^2 V/2\pi\hbar)[N_L N_R/(N_L + N_R)]$; i.e., the resistances of the two point contacts add.

The currents $I_{L,R}$, however, fluctuate as a result of the discreteness of the electron charge. These fast fluctuations (correlated on the time scale $\tau_0 \sim \hbar/eV$) cause slow variations of f_C on the scale of the relaxation time $\tau_C = 2\pi\hbar n_F/(N_L + N_R)$ [21,22]. The cause of this is charge conservation: $I_L + I_R = -\dot{Q}_C$, where $Q_C = e^2 n_F V f_C$ is the charge in the cavity. Fluctuations of f_C in turn affect the currents $I_{L,R}$ thereby generating correlations between them. Our goal is to integrate out currents $I_{L,R}$ taking into account correlations and to obtain the FCS of the transmitted charge $Q(t) = (1/2)\int_0^t dt'[I_R(t') - I_L(t')]$. More precisely, we have to find the generator $S(\chi, t)$ of the charge cumulants defined as

$$P(Q, t) = (2\pi)^{-1} \int d\chi \exp[-i\chi Q + S(\chi, t)], \quad (1)$$

$$\langle Q^m(t) \rangle = \partial^m S(\chi, t)/\partial(i\chi)^m|_{\chi=0}, \quad (2)$$

where $P(Q, t)$ is the distribution of transmitted charge, and $\langle Q^m(t) \rangle$ is its mth cumulant. For $t \to \infty$ one obtains the average current $\langle Q(t) \rangle/t$ and its zero-frequency noise power $\langle Q^2(t) \rangle/t$.

The separation of time scales $\tau_C/\tau_0 \gg 1$ allows us to consider the intermediate time interval Δt, such that $\tau_0 \ll \Delta t \ll \tau_C$, on which the occupation f_C is approximately constant in time. Then the charges transmitted through the point contacts during this time interval, $Q_{L,R} = \int_0^{\Delta t} dt' I_{L,R}$, are not correlated; i.e., the total distribution is a product $P_L(Q_L)P_R(Q_R)$. On the other hand, since Δt is large compared to the correlation time τ_0, their distributions $P_{L,R}$ can be written as Fourier integrals (1) with the cumulant generators proportional to time Δt [23], namely, $S_{L,R} = H_{L,R}\Delta t$. They are well known [2] and for ideal point contacts given by

$$H_{L,R} = (N_{L,R}/2\pi\hbar) \int d\varepsilon \{\ln[1 + f(\varepsilon)(e^{ie\chi_{L,R}} - 1)] + \ln[1 + f_{L,R}(\varepsilon)(e^{-ie\chi_{L,R}} - 1)]\}. \quad (3)$$

For elastic transport, only the energy interval $\mu_R < \varepsilon < \mu_L$ contributes with $f_L = 1$, $f_R = 0$, and $f = f_C$.

To proceed with our derivation we first discretize time, $t_n = n\Delta t$, $n = 1, \ldots, N$, and write the characteristic function of transmitted charge $\exp[S(\chi, t)]$ for the unconditional probability distribution

$$\exp[S(\chi, t)] = \prod_n \prod_{l=L,R} \int dQ_l(t_n)P_l[Q_l(t_n)]$$
$$\times \exp\left\{\frac{i\chi}{2}\sum_n [Q_R(t_n) - Q_L(t_n)]\right\}. \quad (4)$$

Next, we impose a constraint

$$f_C(t_{n+1}) = f_C(t_n) - (e^2 n_F V)^{-1}[Q_L(t_n) + Q_R(t_n)], \quad (5)$$

which guarantees the conservation of charge on the cavity, and gives dynamics to the fluctuating charge Q_C. Integrating out the charges $Q_{L,R}(t_n)$ in Eq. (4) with the use of Eqs. (1) and (5), and taking a continuum limit, we obtain one of our main results

$$\exp[S(\chi, t)] = (2\pi)^{-1} \int \mathcal{D}Q_C \mathcal{D}\chi_C$$
$$\times \exp\left\{\int_0^t dt'[i\chi_C\dot{Q}_C + H(\chi, \chi_C, Q_C)]\right\}, \quad (6)$$

where $H(\chi, \chi_C, Q_C)$ is

$$H = H_L(\chi_C - \chi/2, f_C) + H_R(\chi_C + \chi/2, f_C) \quad (7)$$

with $H_{L,R}$ given by Eq. (3). Equation (6) resembles the imaginary time path integral for the evolution operator of a quantum particle with coordinate Q_C and momentum χ_C, but the Hamiltonian $-H$ given by (7) is not Hermitian. Nevertheless, we can apply the Hamiltonian formalism to our path integral (6). Next, we derive the saddle-point solution and discuss its applicability.

Saddle-point solution.—The variation of the action S in Eq. (6) with respect to χ_C and Q_C gives "classical equations of motion"

$$i\dot{Q}_C = -\partial H/\partial\chi_C, \qquad i\dot{\chi}_C = \partial H/\partial Q_C. \quad (8)$$

These equations are solved with the conditions $Q_C = Q_C(0)$ for $t' = 0$, and $\chi_C = 0$ for $t' = t$ [24]. The solution is substituted into Eq. (6) to obtain the saddle-point part S_0 of the cumulant generator. Equations (8) describe the relaxation of the initial state $Q_C(0)$ to the stationary state $\{\bar{\chi}_C, \bar{Q}_C\}$ given by the solution of $\partial H/\partial\chi_C = \partial H/\partial Q_C = 0$. On time scales large compared to the relaxation time τ_C, the initial and final integration points contribute little (see discussion below), and we can neglect the first term in the action (6). This gives us a large time asymptotic [25],

$$S_0(\chi, t) = tH(\chi, \bar{\chi}_C, \bar{Q}_C), \qquad t \gg \tau_C, \quad (9)$$

which will be analyzed below.

To further simplify the analysis, we concentrate on the symmetric cavity, $N_L = N_R = N_{pc}$. Then the stationary saddle-point solution can be found analytically giving $\bar{\chi}_C = 0$, $\bar{f}_C = 1/2$, and

$$S_0(\chi, t) = (eVN_{pc}t/\pi\hbar)\ln[(1/2)(1 + e^{ie\chi/2})], \quad (10)$$

which is the known result for a symmetric cavity found from quantum mechanical calculations [26].

To investigate the validity of the saddle-point solutions (8) and (9) we calculate the contribution of Gaussian fluctuations by expanding S in the vicinity of the stationary point to second order in $\delta\chi_C$ and δQ_C. For the correction to S_0 we obtain the action of a harmonic oscillator in imaginary time [27] with inverse frequency $\omega^{-1} = \tau_C \cos(e\chi/4)$ and mass $M = (4\pi\hbar/e^3 V N_{pc}) \times \cos^2(e\chi/4)$. In the context of our problem such an action describes the linear dissipative dynamics $\delta\dot{Q}_C = -\omega\delta Q_C + \nu(t)$ with the Gaussian Langevin source $\langle\nu(t)\nu(t')\rangle = M^{-1}\delta(t - t')$ [17], which takes the initial state $Q_C(0) = \bar{Q}_C + \delta Q_C(0)$ to the stationary state $\bar{Q}_C$ after time $t \sim \tau_C$. Integrating out δQ_C we obtain for the total action $S = S_0 + \delta S_0 + S_{fl}$, where $\delta S_0 = -(1/2)M\omega[\delta Q_C(0)]^2 \tanh(\omega t)$ is the initial condition contribution and $S_{fl} = -(1/2)\ln[\cosh(\omega t)]$ is the fluctuation contribution. Since in general $\delta Q_C(0) \lesssim e^2 n_F V$, the part δS_0 is small compared to S_0 if $t \gg \tau_C$. The part S_{fl} is small compared to S_0 if $e n_F V \gg 1$, which has been assumed for our classical cavity [28]. All this justifies the saddle-point solution (9). Having established the framework of our formalism, we now proceed with its generalization and applications.

Generalization.— We consider the nonequilibrium dynamics of an arbitrary classical system, which can be described by fast fluctuating currents $I_{\alpha\beta}$ and slow fluctuating charges Q_α. Metallic reservoirs, such as leads and cavities, can be taken into account by replacing $\alpha \rightarrow (\alpha, l)$, where the index $l = 1, \ldots, L$ enumerates reservoirs, and α enumerates generalized charges in the reservoirs. These charges Q_α are defined as $\dot{Q}_\alpha = -\sum_\beta I_{\alpha\beta}$. Assuming that the probability distributions of fluctuating currents $I_{\alpha\beta}$ are known, we have to find an evolution of the distribution $\Gamma(Q, t)$ of the set of charges $Q = \{Q_\alpha\}$ for a given initial condition $\Gamma(Q, 0)$. In other words, one has to find an evolution operator $U(Q, Q', t)$ such that $\Gamma(Q, t) = \int dQ' U(Q, Q', t)\Gamma(Q', 0)$. We use again the separation of time scales, follow the lines of the derivation of Eq. (6), and obtain the evolution operator $U = \exp[S_0(Q, Q', t)]$, with S_0 being a saddle-point solution of the action

$$S = \int_0^t dt' \left[i\chi\dot{Q} + (1/2)\sum_{\alpha\beta} H_{\alpha\beta}(\chi_\alpha - \chi_\beta) \right], \quad (11)$$

where $\chi = \{\chi_\alpha\}$ is the set of charge counting fields. The generators $H_{\alpha\beta}$ of the cumulants of currents $I_{\alpha\beta}$ have the obvious symmetry $H_{\alpha\beta}(\chi_\alpha - \chi_\beta) = H_{\beta\alpha}(\chi_\beta - \chi_\alpha)$.

Next we assume that the "Hamiltonian" $H = (1/2)\sum_{\alpha\beta}H_{\alpha\beta}$ depends only on the subset of charges $Q^c = \{Q_\alpha^c\}$, which we call conserved charges since they are related to conserved quantities, such as total energy and charge. We complete the set $Q = \{Q^c, Q^a\}$ by the subset of nonconserved or absorbed charges $Q^a = \{Q_\alpha^a\}$. Since H does not depend on Q^a, we find that the corresponding "momenta" $\chi^a = \{\chi_\alpha^a\}$ are integrals of motion: $\chi^a = \text{const}$. This allows us to integrate out the corre-

sponding terms in (11): $i\int_0^t dt' \chi^a \dot{Q}^a = i\chi^a[Q^a(t) - Q^a(0)]$. Thus the rest of the action (11) becomes a generator of cumulants of the absorbed charge $Q^a(t) - Q^a(0)$. In particular, the stationary solution for the cumulant generator is $S_0 = tH(\chi^a, \bar{\chi}^c, \bar{Q}^c)$, where $\bar{\chi}^c$ and $\bar{Q}^c$ are solutions of the equations $\partial H/\partial\chi^c = \partial H/\partial Q^c = 0$.

In the example of the elastic transport considered above there are three reservoirs: two leads ($\alpha = 1, 3$) and one cavity ($\alpha = 2$). The state of the cavity is described by the total charge on the cavity $Q_C = Q_2$ only, while $Q = (Q_3 - Q_1)/2$ is the absorbed charge which is being counted. The counting fields are $\chi_C = \chi_2$ and $\chi/2 = \chi_1 = -\chi_3$. Next we consider the case of inelastic transport requiring the introduction of a new generalized conserved charge E_C, the total energy of the cavity.

Hot-electron regime.—We consider the electron transport through a cavity in the so-called hot-electron regime, i.e., when the electron-electron scattering time is much shorter than the relaxation time $\tau_{ee} \ll \tau_C$, electrons in the cavity then relax to local thermal equilibrium before escaping from the cavity. Their distribution is given by a Fermi function $f(\varepsilon) = \{1 + \exp[(\varepsilon - \mu_C)/T_C]\}^{-1}$ where μ_C and T_C denote, respectively, the electrochemical potential and electron temperature in the cavity. The noise in the hot-electron regime was first considered theoretically for diffusive wires [29]. This calculation and its experimental verification [30] found that electron-electron interaction increases shot noise. Similar results for the chaotic cavity were presented in Ref. [10] and measured in Ref. [31]. We now derive the FCS of transmitted charge Q.

Our calculation starts from the observation that the nonequilibrium state of the cavity is fully described by only two parameters, μ_C and T_C. They can be expressed in terms of conserved values: The total charge $Q_C = e^{-1}C_\mu\mu_C$ and the total energy $E_C = (1/2C_\mu)Q_C^2 + Q_C V_G + (\pi^2/6)n_F T_C^2$ of the cavity, where C_μ is the electrochemical capacitance of the cavity, $1/C_\mu = 1/C + 1/(e^2 n_F)$, and V_G is the gate voltage. The charge and energy conservation can be written as

$$\int d\varepsilon[I_L(\varepsilon) + I_R(\varepsilon)] = -\dot{Q}_C, \quad (12)$$

$$\int d\varepsilon\,\varepsilon[I_L(\varepsilon) + I_R(\varepsilon)] = -e\dot{E}_C, \quad (13)$$

where $I_{L,R}(\varepsilon)$ are the outgoing currents through the point contacts per energy interval $d\varepsilon$. We now replace the constraint (5) with these two equations, and apply it to the energy resolved analog of path integral (4). By doing so, we introduce the counting field χ_E for the total energy E_C. Instead of (7), the new Hamiltonian $H(\chi, \chi_C, \chi_E; \mu_C, T_C)$ takes the form

$$H = \int d\varepsilon[H_L(\chi_C + \varepsilon\chi_E - \chi/2) + H_R(\chi_C + \varepsilon\chi_E + \chi/2)], \quad (14)$$

with $H_{L,R}$ given by Eq. (3). To obtain the FCS in the long time limit $t \gg \tau_C$, where $\tau_C = 2\pi\hbar C_\mu / [e^2 (N_L + N_R)]$ is the RC time (the time of charge screening in the cavity [22]), we need to evaluate H as a function of χ at the saddle point. We find the saddle-point solution $S_0(\chi, t)$ numerically and calculate $P(Q, t)$ by Fourier transformation (1). The result is shown in Fig. 1. It is clearly visible that noise is enhanced in the hot-electron regime.

In order to make further analytical progress we note that the saddle-point solution for $\chi = 0$ is given by the chemical potential $\mu_C = (N_L \mu_L + N_R \mu_R)/(N_L + N_R)$ and the effective temperature $T_C = \sqrt{T^2 + T_V^2}$, where T is the temperature of the reservoirs, and $T_V = (\sqrt{3}/\pi)\eta eV$ is proportional to bias with $\eta^2 = N_L N_R / (N_L + N_R)^2$. The internal fields χ_C, χ_E are zero for $\chi = 0$. It is now straightforward to expand Eq. (14) and the corresponding saddle-point equations around $\chi = 0$ and to calculate higher cumulants order by order. This procedure is very similar to the cascade approach considered in Refs. [13,14]. For the first few cumulants we obtain

$$\langle Q^2 \rangle = \frac{e^2 \eta t}{2\pi\hbar} \sqrt{N_L N_R}(T + T_C), \qquad (15a)$$

$$\langle Q^3 \rangle = -\frac{e^3 \eta^2 t}{2\pi\hbar} \sqrt{N_L N_R} \frac{3\sqrt{3}TT_V}{\pi T_C}, \qquad (15b)$$

$$\langle Q^4 \rangle = \frac{e^4 \eta^3 t}{2\pi\hbar} \sqrt{N_L N_R} \frac{9}{\pi^2} \left[T - T_C + \frac{2T^4}{T_C^3} \right], \qquad (15c)$$

where the second cumulant coincides with Ref. [31]. The result for the third cumulant shows that odd cumulants are strongly suppressed. We note that screening does not affect the FCS in the long time limit $t \gg \tau_C$.

Conclusions.—We have constructed a stochastic path integral formulation of full counting statistics. Our approach is based on the fact that fast microscopic quantum fluctuations give rise to slow variations of conserved quantities. The method has been illustrated with chaotic cavities, and novel results have been presented for the hot-electron regime. We emphasize the general nature of this approach and its applicability to stochastic problems even outside mesoscopic physics.

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